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A COMPARISON BETWEEN THE ADOMIAN DECOMPOSITION AND THE SINC-GALERKIN METHODS FOR SOLVING NONLINEAR BOUNDARY-VALUE PROBLEMS

The second and third authors would like to dedicate this article to the memory of their friend and co-author Elias Deeba who passed away few weeks after the article was submitted for publication

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ABSTRACT

We present a modified Adomian decomposition method for solving nonhomogeneous heat equations and nonlinear ordinary differential equations with boundary conditions and then compare the results with those obtained by using the wavelet-Galerkin and sinc-Galerkin methods.

KEY WORDS: Adomian and modified Adomian decomposition methods, Adomian polynomials, sinc-Galerkin method, non-linear boundary-value problems, non-homogeneous heat equations

1. INTRODUCTION

Many methods are known for solving linear and nonlinear boundary-value problems involving ordinary and partial differential equations, such as the finite differences, finite elements, multi-grid, and Galerkin methods just to mention a few. In the last decade or so, two relatively new techniques, the wavelet-Galerkin and the Adomian decomposition methods, have gained considerable attention. In the wavelet-Galerkin method, the approximate solution is obtained in a multi-resolution analysis setting (see [9, 10]), while in the Adomian decomposition method, which was introduced in [1], the

solution is expressed as a series in which each term is determined from the previous ones using a special algorithm. Numerical implementation of this method has been extensively studied [2, 3, 13, 14].

In a recent paper [6] a comparison was made between the wavelet-Galerkin and sinc-Galerkin methods in solving boundary-value problems involving nonhomogeneous heat equations. It was shown that the sinc-Galerkin method yields better results especially in the presence of singularities. The sinc-Galerkin method, which was introduced by F. Stenger more than twenty years ago [11, 12], is based on the Whittaker-Shannon-Kotel'nikov sampling theorem.

The aim of this paper is 1) to present the modified Adomian decomposition method, introduced in [2], for solving nonhomogeneous heat equations and nonlinear ordinary differential equations with boundary conditions, 2) to compare the results obtained by the Adomian decomposition methods to those obtained using the wavelet-Galerkin and sinc-Galerkin methods for solving boundary-value problems involving nonhomogeneous heat equations, 3) to compare the modified Adomian method with the sinc-Galerkin method for solving boundary-value problems involving nonlinear ordinary differential equations.

The paper is organized as follows. In Sections 2 and 3, we introduce the modified Adomian decomposition method and the sinc-Galerkin method respectively. In Section 4 we compare the wavelet-Galerkin and sinc-Galerkin methods with the Adomian decomposition method for solving boundary-value problems involving nonhomogeneous heat equations, and in Section 5, we compare the sinc-Galerkin method with the modified Adomian method for solving boundary-value problems involving nonlinear ordinary differential equations.

2. THE MODIFIED DECOMPOSITION ALGORITHM

The Adomian decomposition method can be roughly described as obtaining a series solution $u_0 + u_1 + ...$, where each u_i is determined using a special algorithm that we describe below for completeness.
Consider the operator equation

$$Lu + Nu = g, \tag{2.1}$$

where L is a linear operator, N represents nonlinear operator, and g is the known source term.
Assuming that L^{-1} exists and upon applying the inverse operator to both sides of Eq. (2.1), we obtain

$$u = L^{-1}(g) - L^{-1}(Nu). \tag{2.2}$$

The standard Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.3}$$

Under appropriate conditions (e.g. N analytic), the operator N can be decomposed as follows:

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n), \tag{2.4}$$

where A_n are the so-called *Adomian polynomials*.
Substituting this into Eq. (2.2) and for the series to converge, we set

$$\begin{aligned} u_0 &= L^{-1}(g), \\ u_k &= -L^{-1}(A_{k-1}(u_0, u_1, ..., u_{k-1})), \quad k \geq 1. \end{aligned} \tag{2.5}$$

Thus, from Eq. (2.5), we can determine all u's recursively and this defines the standard decomposition method.
For example, if $N(u) = h(u)$ and $h(u)$ is a nonlinear scalar function, we first consider the Taylor expansion of $h(u)$ around u_0 and then collect the terms appropriately to determine A_n. That is,

$$h(u) = h(u_0) + h'(u_0)(u - u_0) + \frac{1}{2!}h''(u_0)(u - u_0)^2 + ... \tag{2.6}$$

Upon substituting the difference $u - u_0$ by the infinite sum into Eq. (2.6), we get

$$h(u) = h(u_0) + h'(u_0)(u_1 + u_2 + ...) + \frac{1}{2!}h''(u_0)(u_1 + ...)^2 + ... \tag{2.7}$$

Adomian polynomials are obtained by reordering and rearranging of the terms of Eq. (2.7). Indeed, to determine the Adomian polynomials, one needs to choose each term in Eq. (2.7) according to the order which actually depends on both the subscripts and the powers of the u_n's.
Therefore, rearranging the terms in the expansion Eq. (2.7) according to the order and assuming that $N(u)$ is as given in Eq. (2.4), then we can give each A_n as

$$\begin{aligned} A_0(u_0) &= h(u_0), \\ A_1(u_0, u_1) &= u_1 h'(u_0), \\ A_2(u_0, u_1, u_2) &= u_2 h'(u_0) + \frac{1}{2!}u_1^2 h''(u_0), \\ A_3(u_0, u_1, u_2, u_3) &= u_3 h'(u_0) + u_1 u_2 h''(u_0) + \frac{1}{3!}u_1^3 h'''(u_0), \\ &..... \end{aligned} \tag{2.8}$$

It is common to note that the decomposition method suggests that the zeroth component u_0 usually defined by the function $L^{-1}(g)$ described above. However, it was shown in [13] that if the function g can be divided into two parts, namely g_1 and g_2, so that the zeroth component u_0 depends upon g_1 while the term u_1 depends upon g_2 and u_0, then this modification leads to a rapid convergence and at times yields an exact solution to the underlying

equation.
The modified algorithm is then represented as:

$$\begin{aligned} u_0 &= L^{-1}(g_1), \\ u_1 &= L^{-1}(g_2) - L^{-1}(A_0(u_0)), \\ u_{k+2} &= -L^{-1}(A_{k+1}(u_0, u_1, ..., u_{k+1})), \quad k \geq 0. \end{aligned} \tag{2.9}$$

As we will see from the examples below, the modified algorithm Eq. (2.9) will require less computation and accelerates the convergence rate. Further, this minor variation in the definition of the components u_0 and u_1 may yield an exact solution by using two iterations only. An important observation that can be made here is that the success of this method depends mainly on the proper choice of the parts g_1 and g_2. The criterion of splitting the function g into two practical parts g_1 and g_2 and using one or the other to define the zeroth term is almost "adhoc" and requires formal analysis. This will be examined in a future study.
In Section 4, we show that the decomposition algorithm is easier to implement for nonhomogeneous heat equations with boundary conditions than other methods.
In Section 5, we observe the efficiency of the modified decomposition algorithm for nonlinear differential equations with boundary conditions. Three nonlinear ordinary differential equations are chosen and the numerical results obtained by using this algorithm are compared with the exact solutions, as well as, with approximate solutions obtained using the sinc-Galerkin method.

3. The Sinc-Galerkin Method

In this section we give a summary of the Sinc-Galerkin Method. The sinc function is defined on the whole real line by

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \qquad -\infty < x < \infty, \tag{3.1}$$

For $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x - kh}{h}\right), \quad \mathrm{k} = 0 \pm 1, \pm 2, ... \tag{3.2}$$

If f is defined on the real line, then for $h > 0$ the series

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(hk) \operatorname{sinc}\left(\frac{x - hk}{h}\right). \tag{3.3}$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of (3.3) has been extensively studied. A comprehensive survey of these approximation properties is found in [11].

To construct approximations on the interval $(0,1)$, which are used in this paper, consider the conformal maps

$$\phi(z) = \ln\left(\frac{z}{1-z}\right). \tag{3.4}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy : \left| arg\left(\frac{z}{1-z}\right)\right| < d \leq \frac{\pi}{2}\right\}, \tag{3.5}$$

onto the infinite strip

$$D_d = \left\{\zeta = \xi + i\eta : |\eta| < d \leq \frac{\pi}{2}\right\} \tag{3.6}$$

The composition

$$S_j(x) = S(h,j) \circ \phi(x) = \text{sinc}\left(\frac{\phi(x) - jh}{h}\right) \tag{3.7}$$

defines the basis element for equation (3.3) on the interval (0,1). The "mesh size" h is the mesh sizes in D_d for the uniform girds $\{kh\}$, $-\infty < k < \infty$. The sinc grid points $z_k \in (0,1)$ in D_E will be denoted by x_k because they are real. The inverse images of the equispaced grids are

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \tag{3.8}$$

Definition 1. *Let D_E be a simply connected domain in the complex plane $\mathbb{C}$, let ∂D_E denote the boundary of D_E. Let $a, b\,(a \neq b)$ be points on ∂D_E, and ϕ be a conformal map D_E onto D_d such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by ψ, define*

$$\Gamma = \{\psi(u) : -\infty < u < \infty\}$$

and $z_k = \psi(kh), \quad k = 0, \pm 1, \pm 2, \ldots$

Definition 2. *Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy*

$$\int_{\psi(L+u)} |F(z)dz| \to 0, \quad as\ u \to \pm\infty, \tag{3.9}$$

where

$$L = \left\{iy : |y| < d \leq \frac{\pi}{2}\right\}, \tag{3.10}$$

and on the boundary of D_E (denoted ∂D_E) satisfy

$$T(F) = \int_{\partial D_E} |F(z)dz| < \infty. \tag{3.11}$$

The importance of the class $B(D_E)$ with regard to numerical integration is summarized in the following theorems [11].

Theorem 3.1. *Let* Γ *be* $(0,1)$*, if* $F \in B(D_E)$ *then for* $h > 0$ *sufficiently small*

$$\int_\Gamma F(z)dz - h\sum_{j=-\infty}^{\infty}\frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D}\frac{F(z)k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F, \tag{3.12}$$

where

$$|k(\phi,h)|\,|_{z\in\partial D} = \left|\exp\left[\frac{i\pi\phi(z)}{h}\ sgn\,(Im\,\phi(z))\right]\right|_{z\in\partial D} = e^{-\pi d/h}. \tag{3.13}$$

For the Sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which exponential convergence results.

Theorem 3.2. *If there exist positive constants* α*,* β *and* C *such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases}\exp\,(-\alpha|\phi(x)|), & x\in\psi((-\infty,0)),\\ \\ \exp\,(-\beta|\phi(x)|), & x\in\psi((0,\infty)).\end{cases} \tag{3.14}$$

then the error bound for the quadrature rule (3.12) is

$$\left|\int_\Gamma F(x)dx - h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Mh}}{\alpha}+\frac{e^{-\beta Nh}}{\beta}\right)+|I_F|\,. \tag{3.15}$$

The infinite sum in (3.12) is truncated with the use of (3.14) to arrive at this inequality (3.15). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \tag{3.16}$$

and

$$N \equiv \left[\left|\frac{\alpha}{\beta}M+1\right|\right], \tag{3.17}$$

where $[x]$ *is the integer part of* x*, then*

$$\int_\Gamma F(x)dx = h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha dM)^{1/2}}\right). \tag{3.18}$$

Theorems 3.1 and 3.2 are used to approximate the integrals that arise in the formulation of the discrete systems corresponding to equations (3.22)-(3.23) below.

To solve a differential equation of the form $Ly = f$ using the sinc-Galerkin method, we assume an approximate solution of the form

$$u_Q(x) = \sum_{j=-M}^{N} c_j S_j(x), \quad Q = M+N+1. \tag{3.19}$$

where $S_j(x)$ is the function $S(j,h)\circ\phi(x)$ for some fixed step size h. The unknown coefficients $\{c_j\}_{-M}^{N}$ in (3.19) are determined by orthogonalizing the residual $Lu_Q - f$ with respect to the functions$\{S_k\}_{k=-M}^{N}$. This yields the discrete system

$$\langle Lu_Q - f, S_k\rangle = 0, \tag{3.20}$$

for $k = -M, -M+1, \ldots, N$. The weighted inner product $\langle\ ,\ \rangle$ is taken to be

$$\langle g(x), f(x)\rangle = \int_0^1 g(x)f(x)w(x)dx, \tag{3.21}$$

Where $w(x)$ plays the role of a weight function which is chosen depending on the boundary conditions, the domain, and the differential equation.

In this paper we will be dealing with nonlinear differential equations of order $2m$, m=1,2,3 of the form:

$$Lu = u^{(2m)} + \tau(x)uu' + \kappa(x)H(u) = f(x), \quad 0 \le x \le 1, \tag{3.22}$$

subject to boundary conditions

$$u^{(j)}(0) = 0, \quad u^{(j)}(1) = 0, \qquad 0 \le j \le m-1 \tag{3.23}$$

where $H(u)$ may be a polynomial or a rational function, or exponential. Due to the large number of different possibilities, our work will be focused mainly on the following forms $H(u)$:

- $H(u) = u^n, \quad n > 1,$
- $H(u) = \exp(\pm\, u).$

We may also include $H(u) = \frac{1}{(1\pm u)^n}, \frac{1}{(1\pm u^2)^n}, \frac{1}{(u^2\pm 1)^n}, \quad n \neq 0$, or $\cos u, \sin u, \cosh u$.., etc or any analytic function of u which has a power series expansion. For the case of boundary value problems of order $2m$, it is convenient to take

$$w(x) = \frac{1}{(\phi'(x))^m}. \tag{3.24}$$

A complete discussion on the choice of the weight function can be found in [8, 12].

The most direct development of the discrete system for equation (3.19) is obtained by substituting (3.19) into (3.22). The system can then be expressed in integral form via (3.21). This approach however, obscures the analysis which is necessary for applying the Sinc Quadrature Formulas to (3.20). An alternative approach is to analyze instead

$$\left\langle u^{(2m)}, S_k\right\rangle + \langle \tau uu', S_k\rangle + \langle \kappa u^n, S_k\rangle = \langle f, S_k\rangle, \qquad k = -M, \ldots, N. \tag{3.25}$$

The method of approximating the integrals in (3.25) begins by integrating by parts to transfer all derivatives from u to S_k. The approximation of the inner products on the right-hand side of (3.25) is

$$\langle f, S_k\rangle = h\frac{f(x_k)w(x_k)}{\phi'(x_k)}. \tag{3.26}$$

Moreover, we have

$$\left\langle u^{(2m)}, S_k \right\rangle = h \sum_{j=-M}^{N} \sum_{i=0}^{2m} \frac{u(x_j)}{\phi'(x_j)h^i} \delta_{kj}^{(i)} g_{2m,i}(x_j), \tag{3.27}$$

and

$$\langle \tau(x)uu', S_k \rangle = -\frac{h}{2} \sum_{j=-M}^{N} \frac{u^2(x_j)}{\phi'(x_j)} \left[\frac{1}{h} \delta_{kj}^{(1)} (\phi'\tau w)(x_j) + \delta_{kj}^{(0)} (\tau w)'(x_j) \right], \tag{3.28}$$

$$\langle \kappa(x)u^n, S_k \rangle = h \frac{w(x_k)u^n(x_k)\kappa(x_k)}{\phi'(x_k)}. \tag{3.29}$$

where

$$\delta_{jk}^{(m)} = h^m \frac{d^m}{d\phi^m} \left[S(j,h) \circ \phi(x) \right]_{x=x_k},$$

and $g_{2m,i}$ are functions to be determined; see [7].

4. EXAMPLES: LINEAR NONHOMOGENEOUS HEAT EQUATIONS

In this section we apply the standard Adomian decomposition method to a linear nonhomogeneous heat equation with boundary conditions. For the sake of comparison with other methods, we choose an example from a paper by El-Gamel and Zayed [6].

Example 1

Consider a nonhomogeneous heat equation with the initial and the boundary conditions.

$$\begin{aligned} u_t - u_{xx} &= g(x,t), \quad 0 \le x \le 1, t > 0, \\ u(0,t) &= 0, \qquad u(1,t) = 0, \\ u(x,0) &= 0, \quad 0 \le x \le 1, \end{aligned} \tag{4.1}$$

where $g(x,t) = [(x - x^2)(1-t) + 2t]e^{-t}$. The exact solution of the equation is $u(x,t) = x(1-x)te^{-t}$.
Before we implement the standard decomposition method, we recognize that Eq. (4.1) is in the equation form of Eq. (2.2) $Lu - u_{xx} = g(x,t)$ with $L = \frac{\partial}{\partial t}$ and $L^{-1}(\bullet) = \int_0^t \bullet ds$. Upon formally integrating Eq. (4.1) with respect to t, we get

$$u(x,t) = \int_0^t \frac{\partial^2 u}{\partial x^2} ds + \int_0^t g(x,s) ds. \tag{4.2}$$

From Eq. (2.5), we have

$$u_0(x,t) = \int_0^t g(x,s)ds = xe^{-t}t - x^2e^{-t}t - 2e^{-t}t - 2e^{-t} + 2, \qquad (4.3)$$

$$u_1(x,t) = \int_0^t \frac{\partial^2 u_0}{\partial x^2}ds = 2e^{-t}t + 2e^{-t} - 2,$$

$$u_2(x,t) = \int_0^t \frac{\partial^2 u_1}{\partial x^2}ds = 0,$$

.....

Hence $u_n(x,t) = 0$ for $\forall n \geq 2$. Therefore, $u(x,t) = u_0(x,t) + u_1(x,t)$ and this is the exact solution. The numerical results in Table 1 for the decomposition method were obtained using Maple.

5. EXAMPLES: NONLINEAR BOUNDARY VALUE PROBLEMS

In this section we apply the standard or modified decomposition methods to boundary-value problems involving nonlinear differential equations. Again for the sake of comparison with the sinc-Galerkin method, we use examples already discussed in [7].

Example 2

Consider a nonlinear ordinary differential equation with the boundary conditions.

$$u^{(4)} - 6e^{-4u} = g(x), \quad 0 < x < 1,$$

and

$$u(0) = 0, \quad u(1) = \ln 2, \quad u'(0) = 1, \quad u'(1) = 0.5$$

where $g(x) = -12(1+x)^{-4}$.
The exact solution of the equation is $u(x) = \ln(1+x)$.

If we integrate the differential equation four-fold with respect to x, we get

$$\begin{aligned} u(x) &= -x + (1 + \frac{\alpha}{2})x^2 + \frac{(\frac{\beta}{2} - 2)x^3}{3} + 2\ln(1+x) \\ &+ 6\int_0^x \int_0^m \int_0^l \int_0^k e^{-4u(s)} dsdkdldm, \end{aligned} \qquad (5.1)$$

which is g the form in Eq.(2.2).
Using Eq. (2.5), we have

$$\begin{aligned} u_0(x) &= -x + 2\ln(1+x) \\ u_1(x) &= (1+\frac{\alpha}{2})x^2 + \frac{(\frac{\beta}{2}-2)x^3}{3} \\ &+ 6\int_0^x\int_0^m\int_0^l\int_0^k (1-4u_0(s))dsdkdldm. \\ u_2(x) &= 6\int_0^x\int_0^m\int_0^l\int_0^k (-4u_1(s)+\frac{32}{2!}u_0(s)u_1(s))dsdkdldm \end{aligned} \quad (5.2)$$

It suffices to compute the first four iterates to get a reasonable error. So, $u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x)$. We use the boundary conditions to obtain α and β. In particular, $\alpha = -1.091225$ and $\beta = 2.467275$.
The numerical results in Table 2 for the decomposition method were obtained using Maple. Note that by only computing the first four iterates, we get comparable results to those of Sinc-Galerkin method discussed in [7] for this example with an error less than 0.01 %.

Example 3

Consider a nonlinear ordinary differential equation with the boundary conditions.

$$\begin{aligned} u^{(6)} + e^{-x}u^2 = g(x), \quad 0 \le x \le 1, \\ u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1 \\ u(1) = 1/e, \quad u'(1) = -1/e, \quad u''(1) = 1/e \end{aligned} \quad (5.3)$$

where $g(x) = e^{-x} + e^{-3x}$.
The exact solution of the equation is $u(x) = e^{-x}$.
Let us consider the numerical solution using the standard decomposition method.
Write Eq. (5.3) in the form of Eq. (2.2). Upon six-fold integration, we get

$$\begin{aligned} u(x) &= \frac{\alpha x^5}{120} + (\frac{\beta}{24} - \frac{1}{36})x^4 + (\frac{\mu}{6} - \frac{5}{27})x^3 \\ &- \frac{1}{54}x^2 + \frac{1}{81}x + e^{-x} + \frac{1}{243}e^{-3x} - \frac{1}{243} \\ &- \int_0^x\int_0^p\int_0^p\int_0^m\int_0^l\int_0^k e^{-s}u(s)^2 dsdkdldmdndp. \end{aligned} \quad (5.4)$$

Using Eq. (2.5), we have

$$\begin{aligned} u_0(x) &= \frac{\alpha x^5}{120} + (\frac{\beta}{24} - \frac{1}{36})x^4 + (\frac{\mu}{6} - \frac{5}{27})x^3 \\ &- \frac{1}{54}x^2 + \frac{1}{81}x + e^{-x} + \frac{1}{243}e^{-3x} - \frac{1}{243} \\ u_1(x) &= -\int_0^x \int_0^p \int_0^p \int_0^m \int_0^l \int_0^k e^{-s} u_0(s)^2 dsdkdldmdndp \\ u_2(x) &= -\int_0^x \int_0^p \int_0^p \int_0^m \int_0^l \int_0^k e^{-s} 2u_0(s)u_1(s) dsdkdldmdndp \end{aligned}$$

It suffices to compute the first two iterates to get a reasonable error. So, $u(x) = u_0(x) + u_1(x)$. We use the boundary conditions to obtain α, β and μ. In particular, $\alpha = .5494856025$, $\beta = .3862454825$, $\mu = -1.001973513$.
The numerical results in Table 3 for the decomposition method were obtained using Maple. Note that by only computing the first two iterates, we get comparable results to those of Sinc-Galerkin method discussed in [7] for this example with an error less than 0.01 %.

Example 4

Consider a nonlinear ordinary differential equation with the boundary conditions.

$$\begin{aligned} u'' + uu' + u^3 = g(x), \quad 0 \le x \le 1, \\ u(0) = 0, \quad u(1) = 0 \end{aligned} \tag{5.5}$$

where $g(x) = \dfrac{1}{x} + x\ln x(1 + \ln x) + (x\ln x)^3$.
The exact solution of the equation is $u(x) = x\ln x$.
Let us consider the numerical solution using the modified decomposition method.
Write Eq. (5.5) in the form of Eq. (2.2). Upon two-fold integration, we get

$$\begin{aligned} u(x) &= \alpha x + x\ln x - x + \frac{1}{6}x^3\ln(x)^2 - \frac{1}{9}x^3\ln x \\ &+ \frac{1}{27}x^3 + \frac{1}{20}x^5\ln(x)^3 - \frac{27}{400}x^5\ln(x)^2 \\ &+ \frac{183}{4000}x^5\ln x - \frac{1107}{80000}x^5 \\ &- \frac{1}{2}\int_0^x u(s)^2 ds - \int_0^x \int_0^l (u(s))^3 dsdl \end{aligned} \tag{5.6}$$

Using Eq. (2.5), we have

$$\begin{aligned} u_0(x) &= (\alpha-1)x + x\ln x \\ u_1(x) &= \frac{1}{6}x^3\ln(x)^2 - \frac{1}{9}x^3\ln x + \frac{1}{27}x^3 + \frac{1}{20}x^5\ln(x)^3 - \frac{27}{400}x^5\ln(x)^2 \\ &+ \frac{183}{4000}x^5\ln x - \frac{1107}{80000}x^5 - \frac{1}{2}\int_0^x u_0(s)^2 ds - \int_0^x\int_0^l (u_0(s))^3 dsdl \\ u_2(x) &= -\frac{1}{2}\int_0^x 2u_0(s)u_1(s)ds - \int_0^x\int_0^l 3u_0(s)^2u_1(s)dsdl \end{aligned} \tag{5.7}$$

It suffices to compute the first three iterates to get a reasonable error. So, $u(x) = u_0(x) + u_1(x) + u_2(x)$. We use the boundary conditions to obtain α. In particular, $\alpha = 0.999999$.

The numerical results in Table 4 for the decomposition method were obtained using Maple. Note that by only computing the first two iterates, we get comparable results to those of Sinc-Galerkin method discussed in [7] for this example with an error less than 0.01 %.

6. CONCLUSION

In this note, we exhibited the Adomian decomposition algorithm Eq. (2.5) and its modified version Eq. (2.9) and showed that, for the examples discussed, these algorithms yield better numerical results and outperform the wavelet-Galerkin method. Although the decomposition algorithms give comparable results to the sinc-Galerkin method, they are easier to implement than the sinc-Galerkin method. Indeed, in the examples discussed, we were able to get the exact solution. For future work, we wish to give the mathematical reasoning behind this algorithm.

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x	Exact	Sinc-Galerkin	Wavelet-Galerkin	A Modified Decomposition
0.0	0.0	0.0	0.0	0.0
$1/2^4$	0.00058	0.00058	0.00051	0.5801073244e-3
$2/2^4$	0.00108	0.00108	0.00096	0.1082867006e-2
$3/2^4$	0.00150	0.00151	0.00133	0.1508279044e-2
$4/2^4$	0.00185	0.00186	0.00166	0.1856343438e-2
$5/2^4$	0.00212	0.00213	0.00190	0.2127060190e-2
$6/2^4$	0.00232	0.00232	0.00208	0.2320429298e-2
$7/2^4$	0.00243	0.00244	0.00218	0.2436450763e-2
$8/2^4$	0.00247	0.00248	0.00222	0.2475124584e-2
$9/2^4$	0.00243	0.00244	0.00218	0.2436450763e-2
$10/2^4$	0.00232	0.00232	0.00208	0.2320429298e-2
$11/2^4$	0.00212	0.00213	0.00190	0.2127060190e-2
$12/2^4$	0.00185	0.00186	0.00166	0.1856343438e-2
$13/2^4$	0.00150	0.00151	0.00136	0.1508279044e-2
$14/2^4$	0.00108	0.00108	0.00099	0.1082867006e-2
$15/2^4$	0.00058	0.00058	0.00052	0.5801073244e-3
1.0	0.0	0.0	0.0	0.0

TABLE 1. Comparison between the Sinc-Galerkin, Wavelet-Galerkin and the Decomposition Methods at $t = 0.01$ (Example 1).

x	Exact	Sinc-Galerkin	A Modified Decomposition
0.0	0.0	0.0	0.0
0.08065	0.077568262040	0.077568262046	.07730684924
0.16488	0.152623517296	0.152623517297	.151725071
0.22851	0.205803507218	0.205803507212	.2043359933
0.39997	0.336452906454	0.336452906455	.333826796
0.5	0.405465108108	0.405465108103	.4027616079
0.69235	0.526121481267	0.526121481263	.524494048
0.77148	0.571819991855	0.571819991858	.57083465
0.88369	0.633234913798	0.633234913793	.63297496
0.94474	0.665133248137	0.665133248135	.665073889
1.0	0.693147180559	0.693147180559	.6931471612

TABLE 2. Comparison between the Sinc-Galerkin and the Modified Decomposition Methods when $\alpha = -1.091225$, $\beta = 2.467275$ (Example 2).

x	Exact	Sinc-Galerkin	A Modified Decomposition
0.0	1.0		1.0
0.0089	0.99113	0.99113	0.9913058120
0.0414	0.95942	0.95942	0.9596798857
0.1721	0.84189	0.84189	0.8420768446
0.3131	0.73113	0.73114	0.7312326771
0.5	0.60653	0.60655	0.6066408944
0.6868	0.50316	0.50320	0.5032981209
0.8278	0.43696	0.43701	0.4372112283
0.9134	0.40114	0.40118	0.4013630017
0.9585	0.38343	0.38347	0.3835873395
1.0	0.36787	0.36787	0.3680302464

TABLE 3. Comparison between the Sinc-Galerkin and the Modified Decomposition Methods when $\alpha = .5494856025$, $\beta = .3862454825$, $\mu = -1.001973513$; (Example 3).

x	Exact	Sinc-Galerkin	A Modified Decomposition
0.0	0.0	0.0	0.0
0.07701	-.19744378	-.19744377	-.1974397778
0.12058	-.25508370	-.25508365	-.2550799778
0.27022	-.35359087	-.35359081	-.3535879602
0.37830	-.36773296	-.36773296	-.3677332289
0.5	-.34657359	-.34657353	-.3465735904
0.62169	-.29549755	-.29549756	-.2954977760
0.72977	-.2298964240	-.22989600	-.2298964241
0.87941	-.11300194	-.11300192	-.1130077475
0.97002	-.02951702	-.02951703	-.02952604034
1.0	0.0	0.0	0.0

TABLE 4. Comparison between the Sinc-Galerkin and the Modified Decomposition Methods when $\alpha = 0.999999$ (Example 4).

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ON THE HYERS-ULAM STABILITY OF AN EULER-LAGRANGE TYPE CUBIC FUNCTIONAL EQUATION *

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ABSTRACT. In this paper, we obtain the general solution and the generalized Hyers-Ulam stability for an Euler-Lagrange type cubic functional equation

$$f(ax+by)+f(ax-by)=ab^2f(x+y)+ab^2f(x-y)+2a(a^2-b^2)f(x)$$

for any fixed integers a, b with $a \neq -1, 0, 1$, $b \neq 0$ and $a \pm b \neq 0$.

1. INTRODUCTION

In 1940, S. M. Ulam [20] gave the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: Under what conditions does there exist a true solution near an approximate function differing slightly from a functional equation? If the answer is affirmative, we say that the functional equation is Hyers-Ulam stable.

During the last decades, the Hyers-Ulam stability problems of several functional equations have been extensively investigated by a number of authors [5, 6, 9, 11, 12, 16, 17]. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds. For more detailed definitions of such terminologies, we can refer to [8, 10, 19].

A quadratic functional equation

$$f(x+y)+f(x-y) \;=\; 2f(x)+2f(y) \tag{1.1}$$

was used to characterize inner product spaces and several other functional equations were also used to characterize inner product spaces ([1], [18]). It is well known that a mapping

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f is a solution of (1.1) if and only if there exists a unique symmetric biadditive mapping B such that $f(x) = B(x, x)$ for all x, where the mapping B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \tag{1.2}$$

Now, we are concerned with the following functional equations, which are related with each other to prove our main subject;

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.3}$$

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y), \tag{1.4}$$

$$\begin{aligned} &f(2x + y) + f(2x - y) + 4f(x) + f(y) + f(-y) \\ &\qquad = 2f(x + y) + 2f(x - y) + 2f(2x), \end{aligned} \tag{1.5}$$

$$\begin{aligned} &f(ax + y) + f(ax - y) \\ &\qquad = af(x + y) + af(x - y) + 2a(a^2 - 1)f(x) \end{aligned} \tag{1.6}$$

for any fixed integer a with $a \neq -1, 0, 1$, and

$$\begin{aligned} &f(ax + by) + f(ax - by) \\ &\qquad = ab^2 f(x + y) + ab^2 f(x - y) + 2a(a^2 - b^2)f(x) \end{aligned} \tag{1.7}$$

for any fixed integers a, b with $a \neq -1, 0, 1$, $b \neq 0$ and $a \pm b \neq 0$. Let both E_1 and E_2 be real vector spaces. The authors [13] proved that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables. They have also investigated the generalized Hyers-Ulam stability problem for the equation (1.3). However it should be noted that (1.3) is a special case of the functional equation (1.7). In [14], the authors showed that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.4) if and only if there exist mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$ and $A : E_1 \to E_2$ such that $f(x) = B(x, x, x) + Q(x, x) + A(x) + f(0)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables, Q is symmetric biadditive and A is additive.

In this paper, we will establish the general solutions of (1.5) and (1.6) which are related with (1.3) and (1.4). Also we are going to solve the generalized Hyers-Ulam stability problem for the equation (1.7) and to extend the results of the generalized Hyers-Ulam stability problem for the equation (1.3).

2. Solutions of (1.5) and (1.6)

Let both E_1 and E_2 be real vector spaces throughout this section. We here present the general solutions of (1.5) and (1.6).

Theorem 2.1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation (1.5) if and only if there exist mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ such that $f(x) = B(x, x, x) + Q(x, x) + A(x)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and is additive for each fixed two variables, Q is symmetric biadditive and A is additive.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.5). Putting $y = x = 0$ in (1.5), we get $f(0) = 0$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$, $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in E_1$. Then $f_e(0) = 0 = f_o(0)$, f_e is even and f_o is odd. Since f is a solution of (1.5), f_e and f_o also satisfy the equation (1.5).

Thus we first assume that f is a solution of the functional equation (1.5) and f is even, $f(0) = 0$. Then the equation (1.5) is written by

$$f(2x + y) + f(2x - y) + 4f(x) + 2f(y) = 2f(x + y) + 2f(x - y) + 2f(2x) \tag{2.1}$$

for all $x, y \in E_1$. Putting $y = x, y = 2x$ in (2.1), separately, we come to

$$f(3x) = 4f(2x) - 7f(x),\ \ f(4x) = 8f(2x) - 16f(x). \tag{2.2}$$

Setting y by $x + y$ in (2.1), one obtains that

$$\begin{aligned} f(3x + y) \ &+ \ f(x - y) + 4f(x) + 2f(x + y) \\ &= \ 2f(2x + y) + 2f(y) + 2f(2x) \end{aligned} \tag{2.3}$$

for all $x, y \in E_1$. Replacing y by $-y$ in (2.3) and adding the resulting relation to (2.3) with use of (2.1), we obtain that

$$f(3x + y) + f(3x - y) + 16f(x) = f(x + y) + f(x - y) + 8f(2x). \tag{2.4}$$

Putting $y = 3x$ in (2.4), we get $f(6x) = 17f(2x) - 32f(x)$.

On the other hand, it follows by (2.2) that

$$f(6x) = 4f(4x) - 7f(2x) = 4[8f(2x) - 16f(x)] - 7f(2x),$$

which yields $f(2x) = 4f(x)$. Therefore the equation (2.4) is now written by

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x). \tag{2.5}$$

Replacing x and y by $\frac{u+v}{2}$ and $\frac{u-v}{2}$ in (2.5), respectively, we obtain that

$$f(2u + v) + f(u + 2v) = 4f(u + v) + f(u) + f(v), \tag{2.6}$$

which implies that $f(x) = Q(x, x)$ for some symmetric biadditive mapping $Q : E_1 \times E_1 \to E_2$ (see [3]).

Next, we may assume that f is a solution of the functional equation (1.5) and f is odd, $f(0) = 0$. Thus the equation (1.5) can be written by

$$f(2x + y) + f(2x - y) + 4f(x) = 2f(x + y) + 2f(x - y) + 2f(2x) \tag{2.7}$$

for all $x, y \in E_1$. Setting x and y by $x + y$ and $x - y$ in (2.7) respectively, we have

$$f(3x + y) + f(x + 3y) + 4f(x + y) = 2f(2x) + 2f(2y) + 2f(2x + 2y). \tag{2.8}$$

Substituting $x + y$ for y in (2.7), we obtain that

$$f(3x + y) + f(x - y) + 4f(x) = 2f(2x + y) - 2f(y) + 2f(2x). \tag{2.9}$$

Switch x with y in (2.9) to get the relation

$$f(x + 3y) - f(x - y) + 4f(y) = 2f(x + 2y) - 2f(x) + 2f(2y). \tag{2.10}$$

Combining (2.9) with (2.10) and using (2.8), one obtains

$$f(2x + 2y) + 3f(x) + 3f(y) = f(2x + y) + f(x + 2y) + 2f(x + y). \tag{2.11}$$

Setting y by $-y$ in (2.11) and then adding it to (2.11), we arrive at

$$\begin{aligned} &f(2x + 2y) + f(2x - 2y) + 10f(x) \\ &\quad = 4f(x + y) + 4f(x - y) + 2f(2x) + f(x + 2y) + f(x - 2y). \end{aligned} \tag{2.12}$$

In turn, substituting $2y$ for y in (2.7), we obtain

$$f(2x + 2y) + f(2x - 2y) + 4f(x) = 2f(x + 2y) + 2f(x - 2y) + 2f(2x). \tag{2.13}$$

Combining (2.12) with (2.13), one obtains that

$$f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y), \tag{2.14}$$

which yields that $f(x) = B(x, x, x) + A(x)$ for all $x \in E_1$ since f is odd and $f(0) = 0$, where B is symmetric for each fixed one variable and additive for each fixed two variables, and A is additive (see [14]).

As a result, we have

$$f(x) = f_e(x) + f_o(x) = B(x, x, x) + Q(x, x) + A(x)$$

for all $x \in E_1$.

Conversely, suppose that there exist mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ such that $f(x) = B(x, x, x) + Q(x, x) + A(x)$ for all $x \in E_1$, where A is additive, Q is symmetric biadditive, and B is symmetric for each fixed one variable and additive for each fixed two variables. Then it is obvious that f satisfies the equation (1.5). □

By virtue of Theorem 2.1 we present the general solution of the functional equation (1.6).

Theorem 2.2. *(i) A mapping $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if (ii) $f : E_1 \to E_2$ satisfies the functional equation (1.6). Therefore, every solution of functional equations (1.3) and (1.6) is a cubic mapping.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.3). Putting $x = 0 = y$ in (1.3), we get $f(0) = 0$. Set $y = 0$ in (1.3) to get $f(2x) = 8f(x)$. Letting $y = x$ and $x = 0$ in (1.3) separately, we obtain that $f(3x) = 9f(x)$ and $f(x) = -f(-x)$ for all $x \in E_1$.

To use an induction argument we assume that for a positive integer $N > 2$, (1.6) is true for any positive integer a with $1 < a \leq N$. Putting y by $x+y$ and y by $x-y$ in (1.6) equipped with $a = N$, separately, we obtain

$$\begin{aligned} &f((N+1)x+y) + f((N-1)x-y) \qquad (2.15)\\ &\qquad = Nf(2x+y) + Nf(-y) + 2N(N^2-1)f(x),\\ &f((N+1)x-y) + f((N-1)x+y) \qquad (2.16)\\ &\qquad = Nf(2x-y) + Nf(y) + 2N(N^2-1)f(x). \end{aligned}$$

Adding (2.15) to (2.16) and using an inductive assumption for $N-1$, we figure out

$$\begin{aligned} &f((N+1)x+y) + f((N+1)x-y) \qquad (2.17)\\ &\quad = (N+1)f(x+y) + (N+1)f(x-y) + 2(N+1)[(N+1)^2-1]f(x), \end{aligned}$$

which proves the validity of (1.6) for $N+1$. Thus the equation (1.6) holds for all positive integer $a > 1$.

For a negative integer $n < -1$, replacing n by $-n > 1$ and using the oddness of f one can easily prove the validity of (1.6).

Therefore the equation (1.3) implies (1.6) for any integer a with $a \neq -1, 0, 1$.

Conversely, let $f : E_1 \to E_2$ satisfy the functional equation (1.6). Putting $x = 0 = y$ and $x = 0$ in (1.6) separately, we get $f(0) = 0$ and $f(y) + f(-y) = 0$. Letting $y = 0$ in (1.6), we obtain $f(ax) = a^3 f(x)$ for all $x \in E_1$. Replacing x and y by $2x$ and ay in (1.6) respectively, we have

$$a^3 f(2x+y) + a^3 f(2x-y) = af(2x+ay) + af(2x-ay) + 2a(a^2-1)f(2x) \qquad (2.18)$$

for all $x, y \in E_1$. Putting y by $x+ay$ in (1.6), we obtain

$$f(a(x+y)+x) + f(a(x-y)-x) = af(2x+ay) + af(-ay) + 2a(a^2-1)f(x). \qquad (2.19)$$

Interchange y and $-y$ in (2.19) to get the relation

$$f(a(x-y)+x) + f(a(x+y)-x) = af(2x-ay) + af(ay) + 2a(a^2-1)f(x). \qquad (2.20)$$

Observe that we get by (1.6)

$$f(a(x+y)+x)+f(a(x+y)-x)=af(2x+y)+af(y)+2a(a^2-1)f(x+y).$$

Adding (2.19) to (2.20), by use of (1.6) we lead to

$$\begin{aligned} &af(2x+y)+2a(a^2-1)f(x+y)+af(2x-y)+2a(a^2-1)f(x-y) \quad (2.21)\\ &=af(2x+ay)+af(2x-ay)+4a(a^2-1)f(x) \end{aligned}$$

for all $x, y \in E_1$. Subtracting (2.21) from (2.18) side by side and dividing by a^3-a, we obtain

$$\begin{aligned} &f(2x+y)+f(2x-y)+4f(x) \quad (2.22)\\ &\qquad =2f(x+y)+2f(x-y)+2f(2x), \end{aligned}$$

which yields by virtue of (2.7) in the proof of Theorem 2.1 that f is cubic since f is odd and $f(ax)=a^3f(x)$ for all $x \in E_1$. That is, f satisfies the equation (1.3). The proof is complete. □

We note that (1.6) implies (1.7). In fact, if $b=\pm 1$ in (1.7), the equation (1.7) reduces (1.6) of itself. Let $b \neq \pm 1$ in (1.7). Then (1.6) implies by the first part of Theorem 2.2

$$f(bx+y)+f(bx-y)=bf(x+y)+bf(x-y)++2b(b^2-1)f(x). \quad (2.23)$$

Setting $y=0$ in (2.23), one gets $f(bx)=b^3f(x)$, and thus $f(\frac{x}{b})=\frac{1}{b^3}f(x)$. Replacing y by by in (2.23) and dividing it by b, we obtain

$$f(x+by)+f(x-by)+2(b^2-1)f(x)=b^2f(x+y)+b^2f(x-y). \quad (2.24)$$

Thus we figure out by (2.24)

$$\begin{aligned} &f(ax+by)+f(ax-by)\\ &\quad =b^3\left[f(a\cdot\frac{x}{b}+y)+f(a\cdot\frac{x}{b}-y)\right]\\ &\quad =ab^3\left[f(\frac{x}{b}+y)+f(\frac{x}{b}-y)+2(a^2-1)f(\frac{x}{b})\right]\\ &\quad =a[f(x+by)+f(x-by)+2(a^2-1)f(x)]\\ &\quad =a[b^2f(x+y)+b^2f(x-y)-2(b^2-1)f(x)+2(a^2-1)f(x)]\\ &\quad =ab^2[f(x+y)+f(x-y)]+2a(a^2-b^2)f(x). \end{aligned}$$

Therefore (1.6) implies (1.7) as desired.

3. Stability of (1.7)

From now on, let X be a topological vector space and let Y be a Banach space unless we give any specific reference. We will investigate the Hyers-Ulam-Rassias stability problem for the functional equation (1.7). Thus we find the condition that there exists a true cubic

mapping near an approximately cubic mapping. For convenience, we use the following abbreviation: for any fixed integers a, b with $a \neq -1, 0, 1$, $b \neq 0$ and $a \pm b \neq 0$

$$\begin{aligned} D_{a,b}f(x,y) := & f(ax+by)+f(ax-by) \\ & -ab^2 f(x+y)-ab^2 f(x-y)-2a(a^2-b^2)f(x) \end{aligned}$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \to \mathbb{R}^+$ be a mapping such that*

$$\sum_{i=0}^{\infty} \frac{\phi(a^i x, 0)}{|a|^{3i}} \qquad \Big(\sum_{i=1}^{\infty} |a|^{3i} \phi(\frac{x}{a^i}, 0), \ \textit{respectively} \Big) \tag{3.1}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(a^n x, a^n y)}{|a|^{3n}} = 0 \qquad \Big(\lim_{n\to\infty} |a|^{3n} \phi(\frac{x}{a^n}, \frac{y}{a^n}) = 0 \Big) \tag{3.2}$$

for all $x,\ y \in X$. Suppose that a mapping $f : X \to Y$ satisfies

$$\|D_{a,b}f(x,y)\| \leq \phi(x,y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T : X \to Y$ which satisfies the equation (1.7) and the inequality

$$\begin{aligned} \|f(x)-T(x)\| &\leq \frac{1}{2|a|^3} \sum_{i=0}^{\infty} \frac{\phi(a^i x, 0)}{|a|^{3i}} \\ \Big(\|f(x)-T(x)\| &\leq \frac{1}{2|a|^3} \sum_{i=1}^{\infty} |a|^{3i} \phi(\frac{x}{a^i}, 0) \Big) \end{aligned} \tag{3.4}$$

for all $x \in X$. The mapping T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}} \qquad \Big(T(x) = \lim_{n\to\infty} a^{3n} f(\frac{x}{a^n}) \Big) \tag{3.5}$$

for all $x \in X$.

Further, if either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

Proof. Putting $y = 0$ in (3.3) and dividing by $2|a|^3$, we have

$$\Big\| \frac{f(ax)}{a^3} - f(x) \Big\| \leq \frac{1}{2|a|^3} \phi(x, 0) \tag{3.6}$$

for all $x \in X$. Replacing x by ax in (3.6) and dividing by $|a|^3$ and summing the resulting inequality with (3.6), we get

$$\Big\| \frac{f(a^2 x)}{a^6} - f(x) \Big\| \leq \frac{1}{2|a|^3} \Big[\phi(x,0) + \frac{\phi(ax,0)}{|a|^3} \Big] \tag{3.7}$$

for all $x \in X$. Using the induction on a positive integer n, we figure out

$$\begin{aligned} \left\| \frac{f(a^n x)}{a^{3n}} - f(x) \right\| &\leq \frac{1}{2|a|^3} \sum_{i=0}^{n-1} \frac{\phi(a^i x, 0)}{|a|^{3i}} \\ &\leq \frac{1}{2|a|^3} \sum_{i=0}^{\infty} \frac{\phi(a^i x, 0)}{|a|^{3i}} \end{aligned} \tag{3.8}$$

for all $x \in X$.

In order to prove the convergence of the sequence $\{\frac{f(a^n x)}{a^{3n}}\}$, we divide inequality (3.8) by $|a|^{3m}$ and also replace x by $a^m x$ to find that for $n,\ m > 0$,

$$\begin{aligned} \left\| \frac{f(a^{n+m}x)}{a^{3n+3m}} - \frac{f(a^m x)}{a^{3m}} \right\| &= \frac{1}{|a|^{3m}} \left\| \frac{f(a^{n+m}x)}{a^{3n}} - f(a^m x) \right\| \\ &\leq \frac{1}{2|a|^3} \sum_{i=0}^{\infty} \frac{\phi(a^i a^m x, 0)}{|a|^{3m+3i}}. \end{aligned} \tag{3.9}$$

Since the right hand side of the inequality tends to 0 as m tends to infinity, the sequence $\{\frac{f(a^n x)}{a^{3n}}\}$ is a Cauchy sequence in Y. Therefore, we may define a mapping $T : X \to Y$ by

$$T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}}$$

for all $x \in X$. By letting $n \to \infty$ in (3.8), we arrive at the formula (3.4).

To show that T satisfies the equation (1.7), replace x and y by $a^n x$ and $a^n y$ in (3.3) respectively, and then divide by $|a|^{3n}$. Then it follows that

$$\begin{aligned} |a|^{-3n} \| f(a^n(ax+by)) &+ f(a^n(ax-by)) - ab^2 f(a^n(x+y)) \\ &- ab^2 f(a^n(x-y)) - 2a(a^2-b^2) f(a^n x)) \| \leq |a|^{-3n} \phi(a^n x, a^n y). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that T satisfies (1.7) for all $x, y \in X$.

To prove the uniqueness of the cubic mapping T subject to (3.4), let us assume that there exists a cubic mapping $S : X \to Y$ which satisfies (1.7) and the inequality (3.4). Obviously, we have $S(a^n x) = a^{3n} S(x)$ and $T(a^n x) = a^{3n} T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.4) that

$$\begin{aligned} \|S(x) - T(x)\| &= |a|^{-3n} \|S(a^n x) - T(a^n x)\| \\ &\leq |a|^{-3n} (\|S(a^n x) - f(a^n x)\| + \|f(a^n x) - T(a^n x)\|) \\ &\leq \frac{1}{|a|^3} \sum_{i=0}^{\infty} \frac{\phi(a^i a^n x, 0)}{|a|^{3n+3i}} \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T.

The proof of assertion indicated by parentheses in the theorem is similarly proved by the following inequality originated from (3.6),

$$\left\| f(x) - a^{3n} f(\frac{x}{a^n}) \right\| \leq \frac{1}{2|a|^3} \sum_{i=1}^{n} |a|^{3i} \phi(\frac{x}{a^i}, 0).$$

In this case, $f(0) = 0$ since $\sum_{i=1}^{\infty} |a|^{3i}\phi(0,0) < \infty$ and so $\phi(0,0) = 0$ by assumption.

The proof of the last assertion in the theorem follows by the same reasoning as the proof of [4]. This completes the proof of the theorem. □

From the main Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability of the equation (1.7). We note that p need not be equal to q.

Corollary 3.2. *Let X and Y be a normed space and a Banach space, respectively, and let ε, p, q be real numbers such that $\varepsilon \geq 0$, $q > 0$ and either $p, q < 3$ or $p, q > 3$. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|D_{a,b}f(x,y)\| \leq \varepsilon(\|x\|^p + \|y\|^q) \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T : X \to Y$ which satisfies the equation (1.7) and the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon\|x\|^p}{2||a|^3 - |a|^p|}$$

for all $x \in X$ and for all $x \in X \setminus \{0\}$ if $p < 0$. The mapping T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}} \quad \text{if } p, q < 3 \qquad \left(T(x) = \lim_{n\to\infty} a^{3n} f(\frac{x}{a^n}) \quad \text{if } p, q > 3\right)$$

for all $x \in X$. If moreover either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

It is significant for us to decrease the possible estimator of the stability problem for the functional equations. This work is possible if we consider the stability problem in the sense of Hyers and Ulam for the functional equation (1.7) with an appropriate large integer a.

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let X and Y be a normed space and a Banach space, respectively, and let $\varepsilon \geq 0$ be a real number. Suppose that a mapping $f : X \to Y$ satisfies*

$$\|D_{a,b}f(x,y)\| \leq \varepsilon \tag{3.11}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}}$ which satisfies the equation (1.7) and the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{2(|a|^3 - 1)} \tag{3.12}$$

for all $x \in X$. Furthermore, if either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

In the last part of this section, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. A cubic mapping $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B- cubic if

$$Q(ax) = a^3Q(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

For a given mapping $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ and a given $u \in B$, we set

$$\begin{aligned} D_{a,b,u}f(x,y) := {} & f(uax + uby) + f(uax - uby) \\ & -u^3ab^2f(x+y) - u^3ab^2f(x-y) - 2u^3a(a^2-b^2)f(x) \end{aligned}$$

for all $x, y \in {}_B\mathbb{B}_1$. We are going to prove the generalized Hyers-Ulam stability problem of the functional equation (1.7) in Banach modules over a unital Banach algebra. As an application of the above Theorem 3.1, we have the following.

Theorem 3.4. *Suppose that a mapping $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$\|D_{a,b,u}f(x,y)\| \le \phi(x,y) \tag{3.13}$$

for all $u \in B$ ($|u| = 1$) and for all $x, y \in {}_B\mathbb{B}_1$ and the mapping $\phi : {}_B\mathbb{B}_1 \times {}_B\mathbb{B}_1 \to \mathbb{R}^+$ satisfies the assumptions of Theorem 3.1.

If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-cubic mapping $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by

$$Q(x) = \lim_{i\to\infty} \frac{f(a^ix)}{a^{3i}} \qquad \Big(Q(x) = \lim_{i\to\infty} a^{3i}f(\frac{x}{a^i})\Big), \tag{3.14}$$

which satisfies the equation (1.7) and the inequality

$$\begin{aligned} \|f(x) - Q(x)\| &\le \frac{1}{2|a|^3}\sum_{i=0}^{\infty}\frac{\phi(a^ix,0)}{|a|^{3i}} \\ \Big(\|f(x) - Q(x)\| &\le \frac{1}{2|a|^3}\sum_{i=1}^{\infty}|a|^{3i}\phi(\frac{x}{a^i},0)\Big) \end{aligned} \tag{3.15}$$

for all $x \in {}_B\mathbb{B}_1$.

Proof. By Theorem 3.1, it follows from the inequality of the statement for $u = 1$ that there exists a unique cubic mapping $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ defined by (3.14) which satisfies the equation (1.7) and inequality (3.15).

Under the assumption that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, the cubic mapping $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$Q(tx) = t^3Q(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, Q is $\mathbb{R}$-cubic.

Replacing x, y by $a^{i-1}x, 0$ in (3.13) respectively, we obtain that for each $u \in B$ ($|u| = 1$)

$$2\|f(ua^ix) - u^3a^3f(a^{i-1}x)\| \le \phi(a^{i-1}x, 0) \tag{3.16}$$

for all $x \in {}_B\mathbb{B}_1$. Using the fact that there exists a positive constant K such that $\|uz\| \leq K|u|\|z\|$ for all $u \in B$ and each $z \in {}_B\mathbb{B}_2$ [2], one can show from (3.16) that

$$\begin{aligned}\|u^3 f(a^i x) - u^3 a^3 f(a^{i-1}x)\| &\leq K|u|^3\|f(a^i x) - a^3 f(a^{i-1}x)\| \\ &\leq K\frac{\phi(a^{i-1}x, 0)}{2}\end{aligned}$$

for all $u \in B(|u| = 1)$ and all $x \in {}_B\mathbb{B}_1$. Thus we get

$$\begin{aligned}&\|f(ua^i x) - u^3 f(a^i x)\| \\ &\leq \left\|f(ua^i x) - u^3 a^3 f(a^{i-1}x)\right\| + \left\|u^3 a^3 f(a^{i-1}x) - u^3 f(a^i x)\right\| \\ &\leq \frac{\phi(a^{i-1}x, 0)}{2} + K\frac{\phi(a^{i-1}x, 0)}{2}\end{aligned}$$

for all $u \in B(|u| = 1)$ and all $x \in {}_B\mathbb{B}_1$. Dividing the above by $|a|^{3i}$ and then taking the limit, we have

$$\begin{aligned}\|Q(ux) - u^3 Q(x)\| &= \lim_{i\to\infty}\left\|\frac{f(ua^i x) - u^3 f(a^i x)}{a^{3i}}\right\| \\ &\leq \lim_{i\to\infty}\frac{\phi(a^{i-1}x, 0) + K\phi(a^{i-1}x, 0)}{2|a|^{3i}} \\ &= 0.\end{aligned}$$

Hence Q satisfies the equation $Q(ux) = u^3 Q(x)$ for all $u \in B(|u| = 1)$ and all $x \in {}_B\mathbb{B}_1$. The last equality is also true for $u = 0$. Since Q is $\mathbb{R}$-cubic and $Q(ux) = u^3 Q(x)$ for each element $u \in B(|u| = 1)$, we figure out

$$\begin{aligned}Q(ax) &= Q(|a| \cdot \frac{a}{|a|}x) = |a|^3 \cdot Q(\frac{a}{|a|}x) = |a|^3 \cdot \frac{a^3}{|a|^3} \cdot Q(x) \\ &= a^3 Q(x)\end{aligned}$$

for all $a \in B(a \neq 0)$ and all $x \in {}_B\mathbb{B}_1$. So the unique $\mathbb{R}$-cubic mapping $Q : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also B-cubic, as desired.

The proof of assertion indicated by parentheses in the theorem is similarly proved. This completes the proof of the theorem. □

Corollary 3.5. *Let E_1 and E_2 be Banach spaces over the complex field $\mathbb{C}$, and let $\varepsilon \geq 0$ be a real number. Suppose that a mapping $f : E_1 \to E_2$ satisfies (3.13) for all $u \in \mathbb{C}$ $(|u| = 1)$ and for all $x, y \in E_1$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-cubic mapping $Q : E_1 \to E_2$ which satisfies the equation (1.7) and the inequality (3.15).*

Proof. Since $\mathbb{C}$ is a Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 3.4, there exists a unique $\mathbb{C}$-cubic mapping $Q : E_1 \to E_2$ satisfying the inequality (3.15). This completes the proof. □

Remark. We ask about the solution and the stability of the following Euler-Lagrange type cubic functional equation

$$f(ax+by)+f(bx+ay) \\ = (a+b)(a-b)^2[f(x)+f(y)]+ab(a+b)f(x+y)$$

for suitable integers a, b with $a \neq 0, b \neq 0, a \pm b \neq 0$.

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q-Hausdorff Summability

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Abstract

We define a q-analog of Cesàro summability and we then construct a class of q-Hausdorff matrices. We define a type of q-difference for sequences and a q-analog of Bernstein polynomials. Using these concepts we define a q-moment problem and relate this moment problem to q-Hausdorff summability.

Keywords: matrix summability, Cesàro summability, Hausdorff matrices, Hausdorff moment problem, Bernstein polynomials, q- binomial theorem.
Math Subject Classification: 40G05,40C05,33D99,33D05.

1 Introduction

If (z_n) is a sequence of complex numbers then the Cesàro mean (σ_n) is defined by

$$\sigma_n = \frac{z_0 + z_1 + ... + z_n}{n+1}, n = 0, 1, 2, ... \tag{1}$$

If $\lim_{n\to\infty} \sigma_n = \sigma$ then the sequence (z_n) is said to be Cesàro summable to the limit σ. It is also said that (z_n) is summable by the Cesàro means of first order, or is summable $(C, 1)$. This is because the Cesàro mean as defined in (1) belongs to a family of summability methods (C, α) where $\alpha \geqslant 0$. We will speak of these more general Cesàro means subsequently. The first order means (1) have played an important role in analysis. Arguably the most famous application of $(C, 1)$ summability is the classic result of L. Fejér in which he proved that the Cesàro means of the Fourier series of a continuous function converge uniformly. This beautiful theorem may be found in most books on Fourier series. The subject of summability methods was a major research topic in the first half of the twentieth century, an excellent reference to this work is provided by G.H. Hardy's classic book Divergent Series [6].

The last thirty years has seen a remarkable production of research involving q-series and q-differences (cf. [5]). This q-analysis has deep roots going back to Euler. The development of the theory of Askey-Wilson polynomials was a primary catalyst in the current interest in the subject. One of the thrusts in this research has been aimed at finding suitable q -analogs of functions and processes belonging to classical function theory. For example in [1] and [3]

first steps were taken in the development of a Fourier theory involving certain q-analogs of trigonometric functions. A complete development of a q-Fourier theory must include a suitable summability theory. In this paper we will take a preliminary step by introducing a q-analog of Cesàro summability and linking it to a q-version of Hausdorff summability.

For the sake of completeness we will make some definitions and fix some notation used in the q-calculus. The standard reference on such things is the book by G. Gasper and M. Rahman [5]. We will always assume that $0 < q < 1$. First, we define the q-coefficient $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$. The infinite version of this product is defined by $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$. The q-binomial coefficient is defined by $\begin{bmatrix} p \\ s \end{bmatrix} = \frac{(q;q)_p}{(q;q)_s(q;q)_{p-s}}$. We will use the notation $[x-a]_q^n = (x-a)(x-aq)...(x-aq^{n-1})$ and throughout the paper we will make frequent use of the finite q-binomial theorem (cf.[5]) which states that

$$[x-a]_q^n = \sum_{j=0}^{n}(-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} a^j x^{n-j}. \tag{2}$$

Lastly, we record the definition of the Jackson q-integral which plays an important role in the q-calculus. If f is a suitably defined function then

$$\int_0^a f(t)d_q t = (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k. \tag{3}$$

We note that the q-integral (3) is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points aq^k, $k = 0, 1, \dots$. The jump at aq^k is $a(1-q)q^k$.

2 q-Cesàro Summability

Let $A = (a_{nk}), n, k = 0, 1, 2, \dots$ be an infinite matrix of real numbers. We will define the A-transform of a given sequence $z = (z_n)$ to be the sequence $t = (t_n)$ defined by

$$t_n = \sum_{k=0}^{\infty} a_{nk} z_k, \quad n = 0, 1, \dots \tag{4}$$

Naturally we presume that the infinite series in (4) converge. The relation (4) can be written in matrix form as $t = Az$. The matrix A is said to be a regular summability method if the convergence of the sequence (z_n) implies the convergence of the transform sequence (t_n) to the same limit. That is, $z_n \to a$ implies that $t_n \to a$. The matrix corresponding to the first order Cesàro means (1) is

$$a_{nk} = \begin{cases} \frac{1}{n+1} & \text{if } k \le n \\ 0 & \text{if } k > n \end{cases} \tag{5}$$

The Silverman-Toeplitz theorem ([6],[8],[9]) provides necessary and sufficient conditions that the matrix A in (4) be regular.

Theorem 1 *(Silverman-Toeplitz): The matrix A is a regular summability method if and only if*

(1) $lim_{n\to\infty} a_{nk} = 0,\ k = 0, 1, \dots,$

(2) $lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk} = 1$,
(3) $\sum_{k=0}^{\infty} |a_{nk}| < M,\ n = 0, 1, \ldots$.

It is obvious that the Cesàro matrix in (5) satisfies the three conditions of Theorem 1. There are many ways to define a q-analog of $(C, 1)$ summability. We will give our suggested analog and then explain why it seems suitable. Define $C_1(q) = (a_{nk}(q))$ where

$$a_{nk}(q) = \begin{cases} \frac{1-q}{1-q^{n+1}} q^{n-k} & \text{if } k \le n \\ 0 & \text{if } k > n \end{cases} \tag{6}$$

We will then say that (z_n) is q-Cesàro summable to the limit a if

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk}(q) z_k = a. \tag{7}$$

The first reason that this definition is appropriate is that $\lim_{q\to 1} a_{nk}(q) = \frac{1}{n+1}$. Thus the q-Cesàro matrix $C_1(q)$ converges to the Cesàro matrix for (C,1) summability as $q \to 1$. Another reason the definition seems appropriate involves the relation between the binomial theorem and the q-binomial theorem. We will explain this now. The Cesàro means of order α satisfy a power series identity that may be taken as their defining relation. Given an infinite series $\sum_{k=0}^{\infty} u_k$, we define the (C, α) mean of the series to be the sequence $(U_n^{(\alpha)})$ in the power series identity

$$(1-z)^{-\alpha-1} \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} U_n^{(\alpha)} z^n, \tag{8}$$

where the numbers $b_n^{(\alpha+1)}$ are the binomial power series coefficients:

$$(1-z)^{-\alpha-1} = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} z^n. \tag{9}$$

If we denote the partial sums of $\sum_{k=0}^{\infty} u_k$ by s_n then the identity (8) is equivalent to

$$(1-z)^{-\alpha} \sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} b_n^{(\alpha+1)} U_n^{(\alpha)} z^n. \tag{10}$$

If we set $\alpha = 1$ in (10) we obtain the $(C, 1)$ mean defined in (1). It seems reasonable to write a q-analog of (9) by using the q-binomial series (cf.[5]).

$$\frac{(q^{\alpha+1}z; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} z^n. \tag{11}$$

If $q \to 1$ in (11) then (9) is obtained. We would then define the q-Cesàro mean of order α of a sequence (u_n) to be the sequence $(U_n^{(\alpha)}(q))$ given by

$$\frac{(q^{\alpha+1}z; q)_\infty}{(z; q)_\infty} \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} U_n^{(\alpha)}(q) z^n. \tag{12}$$

When $\alpha = 1$ in (12) we get the first order q-Cesàro mean as defined in (1) and as defined by the matrix $C_1(q)$. We will denote the summability matrix that

corresponds to $\alpha > 0$ in (12) by $C_\alpha(q)$. Simple calculations establish that the $q-$Cesàro matrix $C_\alpha(q)$ of order α satisfies the conditions of Theorem 1. We thus have

Theorem 2 *The q-Cesàro matrix $C_\alpha(q)$ is a regular summability method if $\alpha > 0$.*

If A and B are summability matrices we say that A is stronger than B if every sequence that is summed by B is also summed by A (to the same limit). If conversely every A summable sequence is also B summable then we say that A and B are equivalent. It is natural to ask how the strength of the first order q-Cesàro means varies with q. The answer is provided in the next theorem.

Theorem 3 *$C_1(q_1)$ and $C_1(q_2)$ are equivalent for $0 < q_1, q_2 < 1$*

Proof. Set $\alpha = 1$ in equation (12) to get

$$\frac{1}{(1-z)(1-qz)} \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q} U_n^{(1)}(q) z^n. \tag{13}$$

If we set $q = q_1$ and $q = q_2$ in (13) we easily find that

$$\frac{1-q_2 z}{1-q_1 z} \sum_{n=0}^{\infty} \frac{1-q_2^{n+1}}{1-q_2} U_n^{(1)}(q_2) z^n = \sum_{n=0}^{\infty} \frac{1-q_1^{n+1}}{1-q_1} U_n^{(1)}(q_1) z^n. \tag{14}$$

Expanding $\frac{1-q_2 z}{1-q_1 z}$ in a power series, multiplying the series on the left of (14), and equating power series coefficients yields

$$U_n^{(1)}(q_1) = \sum_{j=0}^{n} a_{nj} U_j^{(1)}(q_2), \tag{15}$$

where the terms a_{nj} have the form

$$a_{nj} = \begin{cases} (q_1 - q_2) \frac{1-q_2^{j+1}}{1-q_1^{n+1}} \frac{1-q_1}{1-q_2} q_1^{n-j-1} & \text{if } j = 0, 1, ..., n-1 \\ \frac{1-q_2^{n+1}}{1-q_1^{n+1}} \frac{1-q_1}{1-q_2} & \text{if } j = n \end{cases} \tag{16}$$

Equation (16) expresses the sequence $(U_n^{(1)}(q_1))$ as a matrix transform of the sequence $(U_n^{(1)}(q_2))$. A routine calculation shows that the matrix (a_{nk}) satisfies the conditions of Theorem 2. Thus every sequence summable $C_1(q_2)$ is also summable $C_1(q_1)$. To complete the proof, we only need to switch q_1 and q_2 in the calculations above. ■

This theorem does not address the comparison of $C_1(q)$ with the usual Cesàro mean $(C, 1)$. The next theorem deals with this.

Theorem 4 *Any sequence that is summable $C_1(q)$ is also summable $(C, 1)$. The converse statement does not hold.*

Proof. The proof follows the same lines as the proof of Theorem 3. Let (σ_n) denote the $(C, 1)$ mean of a given sequence and let $(U_n(q))$ denote the $C_1(q)$ mean of the same sequence. Then we have $\sigma_n = \sum_{j=0}^{n} \alpha_{nj} U_j(q)$, where

$$\alpha_{nj} = \begin{cases} \frac{1-q^{j+1}}{n+1} & \text{if } j = 0, 1, ..., n-1 \\ \frac{1-q^{n+1}}{(n+1)(1-q)} & \text{if } j = n \end{cases} \tag{17}$$

The matrix (α_{nj}) satisfies the conditions of Theorem 1, hence if $(U_n(q))$ converges then so does (σ_n). To prove the second part of the theorem we write $U_n(q) = \sum_{j=0}^{n} \beta_{nj}\sigma_j$, where

$$\beta_{nj} = \begin{cases} \frac{1-q}{1-q^{n+1}}(j+1)(1-q^{-1})q^n & \text{if } j = 0, 1, ..., n-1 \\ \frac{1-q}{1-q^{n+1}}(n+1) & \text{if } j = n \end{cases} . \tag{18}$$

A calculation shows that $\lim_{n\to\infty} \sum_{j=0}^{n} \beta_{nj} \neq 0$. ■

Consider, for example, the sequence (u_n) defined by $u_n = \frac{1}{2} + cos(x) + cos(2x) + ... + cos(nx)$. It is well known that (u_n) is $(C, 1)$ summable to 0 provided $x \neq 2k\pi$. However, it is not $C_1(q)$ summable.

Remark: The q-Cesàro matrix $C_1(q)$ appears in the Pólya-Szegő problem book [7], and in [4]. However neither of these references have placed $C_1(q)$ in the context of Hausdorff summability as will be done here.

3 Hausdorff Summability

The Cesàro means (C, α) belong to an important class of summability methods called Hausdorff Methods. We will give a very brief outline of the subject here. We will follow the development in [8], other presentations may be found in [6] and [9]. Let C denote the matrix that corresponds to $(C, 1)$ summability. We seek a matrix H with the property that $HC = DH$ where D is diagonal. Solving the matrix equation we find that $H = (h_{pq})$ with

$$h_{pq} = (-1)^{p-q}\binom{p}{q}h_{pp}. \tag{19}$$

The numbers h_{pp} are arbitrary as long as they are non-zero. We choose $h_{pp} = (-1)^p$ and then the matrix H has elements given by

$$h_{pq} = (-1)^q\binom{p}{q}. \tag{20}$$

The matrix H is self-inverse, that is, $H^{-1} = H$. The diagonal matrix D has diagonal elements $d_p = \frac{1}{p+1}$. With these matrices we have $C = H^{-1}DH$. Now we define a Hausdorff matrix to be of the form $A = H^{-1}DH$ where H is the matrix with elements as in (20) and D is any diagonal matrix. Thus Hausdorff matrices can be viewed as generalizations of $(C, 1)$ summability. We need three fundamental theorems pertaining to Hausdorff matrices.

Theorem 5 *A triangular matrix A commutes with C (the $(C, 1)$ matrix) if and only if A is a Hausdorff matrix.*

Theorem 6 *A Hausdorff matrix $H^{-1}DH$ is regular if and only if $D = (d_p\delta_{pq})$ with*

$$d_p = \int_0^1 t^p d\phi(t), \ p = 0, 1, \ldots \tag{21}$$

where the function $\phi(t)$ is of bounded variation on $[0, 1]$, $\phi(1) - \phi(0) = 1$, and $\phi(0^+) = \phi(0)$.

A sequence that has the integral form above is called a Hausdorff moment sequence. It is important to record a formula for the elements of a Hausdorff matrix. Given a sequence (d_p) we define the k^{th} forward difference by

$$\Delta^k d_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} d_{n+m}. \tag{22}$$

We define the k^{th} backward difference by

$$\nabla^k d_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} d_{n+k-m}. \tag{23}$$

The backward and forward differences clearly satisfy the identity $\Delta^k d_n = (-1)^k \nabla^k d_n$. Now if $\Lambda = (\lambda_{km})$ is a Hausdorff matrix $\Lambda = H^{-1}DH$ with $D = (d_p \delta_{pq})$ then

$$\lambda_{km} = \binom{k}{m} \Delta^{k-m} d_m. \tag{24}$$

Theorem 7 *The sequence (d_p) has the form*

$$d_p = \int_0^1 t^p d\phi(t), \ p = 0, 1, \ldots \tag{25}$$

if and only if

$$(-1)^k \Delta^k d_n \geq 0, n, k = 0, 1, \ldots \tag{26}$$

4 q-Hausdorff Summability

In this section we will parallel the connections between $(C, 1)$ and Hausdorff means for the case of q-Cesàro and a q-analog of Hausdorff matrices. We begin by finding a matrix H_q that plays the role of the self-inverse matrix H given by (19).

Theorem 8 *If D is a diagonal matrix then the matrix equation $H_q C_1(q) = D H_q$ has solution $H_q = (h_{ps})$ with*

$$h_{ps} = (-1)^{p-s} \begin{bmatrix} p \\ s \end{bmatrix} h_{pp} q^{(s^2 - s - p^2 + p)/2}, \ s = 0, 1, \ldots p \tag{27}$$

The diagonal matrix D is given by $D = (d_p \delta_{ps})$ with

$$d_p = \frac{1-q}{1-q^{p+1}}. \tag{28}$$

Proof. The proof is a standard matrix calculation. ■

The terms h_{pp} in (27) are arbitrary as long as they are non-zero. Accordingly, taking $h_{pp} = (-1)^p$, the matrix H_q is found to be given by

$$h_{ps} = (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{(s^2 - s - p^2 + p)/2}, \ s = 0, 1, \ldots, p. \tag{29}$$

The matrix H_q is not self-inverse as is the case with the matrix H that was defined in (20). It is easy however to compute the inverse and we find $H_q^{-1} = (h_{ps}^*)$ where

$$h_{ps}^* = h_{ps} q^{(p-s)(p-s-1)/2}. \tag{30}$$

It should be noted that the sequence defined in (28) is a Hausdorff moment sequence and hence the q-Cesàro matrix is a Hausdorff matrix. This is seen by writing

$$d_p = (1-q)\sum_{k=0}^{\infty} q^{kp} q^k = \int_0^1 t^p d_q t, \tag{31}$$

and recalling that the q-integral is a Riemann-Stieltjes integral. The more general q-Cesàro matrix of order α defined by (2.8) also involves a moment sequence. To see this we denote the matrix by $C_\alpha(q) = (a_{n,k})$ and note that $a_{n,n} = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}$. Now we appeal to Lemma 2.1 in [3] which states:

Lemma 1 *If 0<b<a<1 then*

$$\frac{(a;q)_k}{(b;q)_k} = \int_0^1 t^k d\Psi(t) \tag{32}$$

where $\Psi(t)$ is a monotone increasing step function.

We can thus conclude that if $\alpha > 0$ then the general q-Cesàro matrix is a Hausdorff matrix. We now define a q-Hausdorff matrix to be a lower triangular matrix of the form $H_q^{-1} D H_q$ where D is a diagonal matrix. Thus as $q \to 1$ a q-Hausdorff matrix $H_q^{-1} D H_q$ approaches a Hausdorff matrix HDH.
Next, the form of the matrix elements in a q-Hausdorff matrix will be determined.

Definition 1 *For a given sequence (d_p) we define the k^{th} forward q-difference of (d_p) by*

$$\Delta_q^{(k)} d_p = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{(k-j)(k-j-1)}{2}} d_{j+p}, k = 0, 1, \ldots \tag{33}$$

We define the k^{th} backward q-difference by

$$\nabla_q^{(k)} d_p = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} d_{k+p-j}. \tag{34}$$

Note that as $q \to 1$ the forward q-difference approaches the standard forward difference defined in (22) and the backward q-difference approaches the backward difference in (23). Also, we have the identity $\Delta_q^{(k)} d_s = (-1)^k \nabla_q^{(k)} d_s$. A matrix calculation shows that we have:

$$H_q^{-1} D H_q = (\lambda_{ps}),\ \lambda_{ps} = (-1)^s h_{ps} \Delta_q^{(p-s)} d_p = (-1)^p h_{ps} \nabla_q^{(p-s)} d_p, \quad s = 0, 1, ..., p;\ p = 0, 1, \ldots \tag{35}$$

The forward difference defined by (22) satisfies the identity

$$\Delta^n d_p = \Delta^{n-1} d_p - \Delta^{n-1} d_{p+1} \tag{36}$$

The forward q-difference defined by (33) satisfies a similar identity as we prove next.

Theorem 9 *The forward q-difference defined in (33) satisfies the identity*

$$\Delta_q^{(n)} d_s = q^{n-1}\Delta_q^{(n-1)} d_s - \Delta_q^{(n-1)} d_{s+1}. \tag{37}$$

Proof. Use the identity $\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$ to write

$$\Delta_q^{(n)} d_s = \sum_{j=0}^{n-1} (-1)^j q^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{\frac{(n-j)(n-j-1)}{2}} d_{j+s} -$$
$$- \sum_{j=0}^{n-1} (-1)^j \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{\frac{(n-j-1)(n-j-2)}{2}} d_{j+s+1}.$$

A simple rearrangement of the sums gives (37). ■
The identity (37) written in terms of the backward difference becomes

$$\nabla_q^{(n)} d_p = \nabla_q^{(n-1)} d_p - q^{n-1}\nabla_q^{(n-1)} d_{p+1}. \tag{38}$$

5 A Class of q-Hausdorff Matrices

The q-Cesàro matrix $C_1(q) = H_q^{-1} D H_q$ is generated by the moment sequence $d_p = \int_0^1 t^p d_q t$. In this section, a class of q-Hausdorff matrices that generalize $C_1(q)$ will be introduced. Given a sequence of positive numbers a_k with $a_0 = 1$, $a_{k+1} < a_k, k = 0, 1, \ldots$, and $a_k \to 0$. Define a function $\Psi_q(t)$ by $\Psi_q(t) = a_k - a_{k+1}, q^k \le t < q^{k-1}, k = 1, 2, \ldots, \Psi_q(0) = 0, \Psi_q(t) = 1,\ t \ge 1$. For each such sequence and each such resulting weight function $\Psi(t)$ we have a q-Hausdorff matrix where the diagonal matrix D has entries given by

$$d_p = \int_0^1 t^p d\Psi_q(t). \tag{39}$$

In particular when $a_k = q^k$ then $d\Psi_q(t) = d_q t$ and the q-Hausdorff matrix is $C_1(q)$.

Theorem 10 *The matrices $H_q^{-1} D H_q$ where the elements of D are given by (39) are regular.*

Proof. We must show that if d_p is given by (39) then the matrix elements λ_{ps} given by (34) satisfy the three conditions of Theorem 2. We will consider the three conditions in order.

(i) To prove that $\lambda_{ps} \to 0$ as $p \to \infty$ for each $s = 0, 1, \ldots$ we must first compute the difference $\nabla_q^{(p-s)} d_s$. We have

$$\begin{gathered} \nabla_q^{(p-s)} d_s = \textstyle\sum_{j=0}^{p-s} (-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} d_{p-j} \\ = \textstyle\int_0^1 \sum_{j=0}^{p-s} (-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} t^{p-j} d\Psi_q(t) = \int_0^1 t^s [t-1]_q^{p-s} d\Psi_q(t). \end{gathered} \tag{40}$$

Note that $[t-1]_q^{p-s} = (t-1)(t-q)\ldots(t-q^{p-s-1}) = 0$ when $t = q^m$, $m = 0, 1, \ldots p-s-1$. Thus

$$\nabla_q^{(p-s)} d_s = \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi_q(t). \tag{41}$$

After some calculations, it is found that

$$\left|\nabla_q^{(p-s)} d_s\right| \leq q^{\frac{(p-s)(p-s-1)}{2}} (q;q)_{p-s} q^{(p-s)s} q^{p-s} \left[\Psi_q(q^{p-s}) - \Psi_q(0)\right]. \tag{42}$$

Thus we have $|\lambda_{ps}| \leq \frac{(q;q)_p}{(q;q)_s} q^{p-s}$. This proves that $\lambda_{ps} \to 0$ as $p \to \infty$ for fixed s.

(ii) Here, it will be proven that $\lim_{p\to\infty} \sum_{s=0}^{p} \lambda_{ps} = 1$. From (34) and from (39) we get

$$\sum_{s=0}^{p} \lambda_{ps} = (-1)^p q^{-\frac{p(p-1)}{2}} \int_0^1 \sum_{s=0}^{p} (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^s [t-1]_q^{p-s} d\Psi_q(t). \tag{43}$$

In the right side of (43) use the expansion
$[t-1]_q^{p-s} = \sum_{j=0}^{p-s} (-1)^j \begin{bmatrix} p-s \\ j \end{bmatrix} q^{\frac{j(j-1)}{2}} t^{p-s-j}$, and use the identity $\begin{bmatrix} p \\ s \end{bmatrix} \begin{bmatrix} p-s \\ j \end{bmatrix} = \begin{bmatrix} p-j \\ s \end{bmatrix} \begin{bmatrix} p \\ j \end{bmatrix}$, and interchange the sums to get

$$\begin{array}{l} \int_0^1 \sum_{s=0}^{p} (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^s [t-1]_q^{p-s} d\Psi_q(t) = \\ \int_0^1 \sum_{j=0}^{p} \begin{bmatrix} p \\ j \end{bmatrix} (-1)^j q^{\frac{j(j-1)}{2}} \sum_{s=0}^{p-j} (-1)^s \begin{bmatrix} p-j \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^{p-j} d\Psi_q(t). \end{array} \tag{44}$$

Note that $\sum_{s=0}^{p-j} (-1)^s \begin{bmatrix} p-j \\ s \end{bmatrix} q^{\frac{s(s-1)}{2}} t^{p-j} = \delta_{pj}$, and thus the right side of (44) reduces to $(-1)^p q^{\frac{p(p-1)}{2}} \int_0^1 d\Psi_q(t)$. Thus we have

$$\sum_{s=0}^{p} \lambda_{ps} = \int_0^1 d\Psi_q(t) = 1. \tag{45}$$

(iii) Here we must prove that $\sum_{s=0}^{p} |\lambda_{ps}|$ is uniformly bounded. But it is easy to use an argument like that in (i) to see that $\lambda_{ps} \geq 0$, the bound then follows from (ii). ■

As a further example of such a q-Hausdorff matrix we discuss a q-analog of Euler summability (cf.[6]). Here we will take the q-Hausdorff matrix to have elements

$$\lambda_{ps} = \frac{\begin{bmatrix} p \\ s \end{bmatrix} q^{(p-s)(p-s-1)/2} a^{p-s} x^s}{[x+a]_q^p}, 0 < a < x. \tag{46}$$

A calculation shows that the associated diagonal matrix has elements given by

$$d_p = \frac{1}{(-\frac{a}{x};q)_p}. \tag{47}$$

Write $\alpha = \frac{a}{x}$, we have $0 < \alpha < 1$. We can then write

$$d_p = \frac{(-\alpha q^p;q)_\infty}{(-\alpha;q)_\infty} = \frac{1}{(-\alpha;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \alpha^n q^{np}}{(q;q)_n}. \tag{48}$$

The right side of (48) is a Riemann-Stieltjes integral of the form (39) in which the weight function $\Psi(t)$ has jumps at the points q^n and the jump $j(q^n)$ at q^n has value

$$j(q^n) = \frac{q^{\binom{n}{2}} \alpha^n}{(q;q)_n (-\alpha;q)_\infty}. \tag{49}$$

We note that when $q \to 1$ the matrix elements in (46) approach the matrix elements for Euler summability.

The examples of q-Hausdorff summability shown here all have weight functions that are purely discrete and have jumps at the points q^j, the resulting Riemann-Stieltjes integrals thus are all very similar to the Jackson q-integral. In the next section it will be shown that this is not accidental.

6 Relation to the Hausdorff Moment Problem

It is known that a Hausdorff matrix HDH is regular if and only if the sequence that forms the main diagonal in D is a Hausdorff moment sequence ([6], [8], [9]). We will now form a similar connection for a q-Hausdorff matrix. We will say that a sequence (d_p) is totally q-monotone if $\Delta_q^{(n)} d_p \geq 0, n, p = 0, 1, \ldots$. We define a class of weight functions F as follows.

Definition 2 *$\alpha(t)$ belongs to the class F if $\alpha(t)$ is bounded and monotone increasing with jumps at the points q^j, $j = 0, 1, \ldots$, $\alpha(0) = 0$, and if $\alpha(t)$ has no other point of increase.*

Theorem 11 *(d_p) is totally q-monotone if and only if $d_p = \int_0^1 t^p d\Psi(t)$, where $\Psi(t) \in F$.*

Proof. First, suppose that d_p is of the form stated with $\Psi(t) \in F$. We compute the q-difference and find that if $a_j > 0$ is the jump at q^j then

$$\begin{aligned} \Delta_q^{(k)} d_s &= \int_0^1 (1-t)(q-t)(q^2-t)\ldots(q^{k-1}-t)t^s d\Psi(t) \\ &= \sum_{j=k}^{\infty} (1-q^j)(q-q^j)\ldots(q^{k-1}-q^j)a_j q^{js} > 0 \end{aligned}$$

In the other direction the proof follows the lines of the presentation given by Wall [8], the original idea of the proof is due to Schoenberg . We begin with the observation that if $\Delta_q^{(n)} d_s \geq 0, n, s = 0, 1, \ldots$ then for any integer p we have

$$\begin{array}{cc} d_n \geq 0, & n = 0, 1, ..., p \\ \Delta_q^{(1)} d_n \geq 0, & n = 0, 1, ..., p-1 \\ \cdots & \\ \Delta_q^{(p-1)} d_n \geq 0, & n = 0, 1 \\ \Delta_q^{(p)} d_n \geq 0, & n = 0 \end{array} \tag{50}$$

From (37) it follows that the above equations are equivalent to the inequalities

$$\begin{array}{c} \Delta_q^{(p)} d_0 \geq 0 \\ \Delta_q^{(p-1)} d_1 \geq 0 \\ \cdots \\ \Delta_q^{(1)} d_{p-1} \geq 0 \\ \Delta_q^{(0)} d_n \geq 0 \end{array} \tag{51}$$

If we define $r_{p,n} = \Delta_q^{(p-n)} d_n$ the system (51) can be written using (33) as

$$r_{p,n} = \sum_{m=0}^{p} (-1)^{m-n} \begin{bmatrix} p-n \\ m-n \end{bmatrix} d_m q^{\frac{(p-m)(p-m-1)}{2}}, \ n = 0, 1, \ldots, p. \tag{52}$$

Note that the terms in the sum in (52) vanish if $m \leq n-1$. The system of equations (52) can be solved for d_m, the result is

$$d_m = \sum_{k=0}^{p} \begin{bmatrix} p-m \\ p-k \end{bmatrix} q^{m(p-k)} r_{p,k} q^{\frac{k(k-1)-p(p-1)}{2}}. \tag{53}$$

Again, the terms in the above sum vanish if $k \leq m-1$. Define $L_{p,k} = \begin{bmatrix} p \\ k \end{bmatrix} r_{p,k} q^{\frac{k(k-1)-p(p-1)}{2}}$, and use this definition in (53) to get

$$d_m = \sum_{k=0}^{p} \frac{\begin{bmatrix} p-m \\ p-k \end{bmatrix}}{\begin{bmatrix} p \\ k \end{bmatrix}} q^{m(p-k)} L_{p,k}. \tag{54}$$

Note that

$$\frac{\begin{bmatrix} p-m \\ p-k \end{bmatrix}}{\begin{bmatrix} p \\ k \end{bmatrix}} = \frac{(q^{k-m+1};q)_m}{(q^{p-m+1};q)_m} \tag{55}$$

which yields

$$\begin{aligned} d_m &= \textstyle\sum_{k=0}^{p} \frac{(q^{k-m+1};q)_m}{(q^{p-m+1};q)_m} q^{m(p-k)} L_{p,k} \\ &= \textstyle\sum_{k=0}^{p} \frac{\left[q^{p-k}-q^{p-m+1}\right]_q^m}{(q^{p-m+1};q)_m} L_{p,k} \end{aligned} \tag{56}$$

Now make a change of index $j = p-k$ in (56) and write $B_{p,j} = L_{p,p-j}$ to finally obtain

$$d_m = \frac{1}{(q^{p-m+1};q)_m} \sum_{j=0}^{p} \left[q^j - q^{p-m+1}\right]_q^m B_{p,j}. \tag{57}$$

The sum on the right side of (57) represents the evaluation of a Riemann-Stieltjes integral with jumps at the points q^j, $j = 0, 1, \ldots, p$, the jump at each such point is $B_{p,j}$. If we define the step function $\Lambda_p(t)$ by

$$\Lambda_p(t) = \begin{cases} 0, & t < q^p \\ B_{p,p}, & q^p \leq t < q^{p-1} \\ B_{p,p} + B_{p,p-1}, & q^{p-1} \leq t < q^{p-2} \\ \cdots & \\ B_{p,0} + B_{p,1} + \ldots + B_{p,p-1} + B_{p,p}, & 1 \leq t \end{cases} \tag{58}$$

then we may write equation (57) in the form

$$d_m = \frac{1}{(q^{p-m+1};q)_m} \int_0^1 [t - q^{p-m+1}]_q^m d\Lambda_p(t). \tag{59}$$

Note that the function $\Lambda_p(t)$ is bounded because it is monotone increasing and $\Lambda_p(1) = d_0$ from (53). Now observe that

$$\frac{1}{(q^{p-m+1};q)_m} = 1 + q^p O(1) \text{ as } p \to \infty. \tag{60}$$

Also,

$$[t-q^{p-m+1}]_q^m=\sum_{j=0}^{m}\begin{bmatrix}p\\ j\end{bmatrix}(-1)^j q^{\frac{j(j-1)}{2}}q^{(p-m+1)j}t^{m-j}=t^m+q^pO(1),\ \text{as } p\to\infty. \tag{61}$$

Equation (59) can thus be written as

$$d_m=\int_0^1 t^m d\Lambda_p(t)+q^pO(1). \tag{62}$$

We can now apply the Helly-Bray Selection Theorem (cf.[9]) to (62) and allowing $p\to\infty$, the existence of a bounded and non-decreasing function $\Lambda(t)$ such that

$$d_m=\int_0^1 t^m d\Lambda(t) \tag{63}$$

is established. Further, since each function $\Lambda_p(t)$ has jumps at $1,q,q^2,\dots q^p$, and $\Lambda_p(0)=0$ it follows that the limit function $\Lambda(t)$ has jumps at $q^j, j=0,1,2,\dots,$ and that $\Lambda(0)=0$. Thus $\Lambda(t)\in F$. This proves the theorem. ■
We now need some lemmas. The proofs are direct and we only outline one proof.

Lemma 2 $x^n=\sum_{k=0}^n \begin{bmatrix}n\\ k\end{bmatrix}[x-1]_q^k,\ n=0,1,\dots$

Definition 3 *Let $\Lambda_{ps}[x]$ be the polynomial of degree p defined by*

$$\Lambda_{ps}[x]=(-1)^p h_{ps}x^s[x-1]_q^{p-s}. \tag{64}$$

Also, for a given sequence (d_n) define a linear functional M acting on polynomials by $M(x^n)=d_n$.

A calculation shows that $M[\Lambda_{ps}[x]]=\lambda_{ps}$. We will make use of the following identity that has a straightforward induction proof, which is omited.

Lemma 3 *If $0\le n\le p$ then*

$$x^n=\sum_{s=n}^{p}\frac{\begin{bmatrix}s\\ n\end{bmatrix}}{\begin{bmatrix}p\\ n\end{bmatrix}}q^{n(p-s)}\Lambda_{ps}[x]. \tag{65}$$

Next, for a function f defined on the points q^k define the q-Bernstein polynomial associated with f to be

$$B_p[f[x]]=\sum_{s=0}^{p}f(q^{p-s})\Lambda_{ps}[x]. \tag{66}$$

Lemma 4 *If $0\le n\le s\le p$, then $\left\{\frac{\begin{bmatrix}s\\ n\end{bmatrix}}{\begin{bmatrix}p\\ n\end{bmatrix}}-1\right\}q^{p-s}=q^pO(1)$ as $p\to\infty$.*

Proof. The integer n is considered to be fixed. We have

$$\frac{\begin{bmatrix}s\\ n\end{bmatrix}}{\begin{bmatrix}p\\ n\end{bmatrix}}=\frac{(q^{s-n+1};q)_n}{(q^{p-n+1};q)_n}. \tag{67}$$

Also, $(q^{s-n+1};q)_n = \sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} q^{(s-n+1)j} = 1 + q^s O(1)$ as $s \to \infty$. Using the q-binomial theorem we have

$$\frac{1}{(q^{p-n+1};q)_n} = \frac{(q^{p+1};q)_\infty}{(q^{p-n+1};q)_\infty} = \sum_{j=0}^{\infty} \frac{(q^n;q)_j}{(q;q)_j} q^{(p-n+1)j} = 1 + q^p O(1). \tag{68}$$

Using these expressions we get the result. ■

Lemma 5 *If* $\sum_{s=0}^{p} |\lambda_{ps}| < K$ *for* $p = 0, 1, \ldots$ *then* $\lim_{p\to\infty} M[B_p[x^n]] = d_n$.

Proof. We have $B_p[x^n] = \sum_{s=0}^{p} q^{n(p-s)} \Lambda_{ps}[x]$ and consequently $M[B_p[x^n]] = \sum_{s=0}^{p} q^{n(p-s)} \lambda_{ps}$. From Lemma 4 recalling that $M[x^n] = d_n$ and applying M on both sides of (65) we get

$$d_n = \sum_{s=n}^{p} \frac{\begin{bmatrix} s \\ n \end{bmatrix}}{\begin{bmatrix} p \\ n \end{bmatrix}} q^{n(p-s)} \lambda_{ps}, \tag{69}$$

thus we may write

$$d_n - M[B_p[x^n]] = \sum_{s=n}^{p} \left\{ \frac{\begin{bmatrix} s \\ n \end{bmatrix}}{\begin{bmatrix} p \\ n \end{bmatrix}} - 1 \right\} q^{n(p-s)} \lambda_{ps} - \sum_{s=0}^{n} q^{n(p-s)} \lambda_{ps}. \tag{70}$$

Note that the right side of the above expression vanishes when $n = 0$ and the lemma then holds trivially. We may then suppose that $n \geq 1$ for the remainder of the proof. The second sum on he right of (70) is of the form $q^p O(1)$ as $p \to \infty$.The first sum also has that form by Lemma (4). This proves the result. ■

Definition 4 $\alpha(t) \in F^*$ *if* $\alpha(t)$ *has points of increase at* $q^k, k = 0, 1, \ldots$ *and nowhere else,* $\alpha(0) = 0$, *and if* $\alpha(t)$ *is of bounded variation on* $[0, 1]$.

Theorem 12 *A q-Hausdorff matrix is regular if and only if* d_m *is given by (63) with* $\Lambda(t) \in F^*$.

Proof. If d_m is given by (63) with $\Lambda(t) \in F^*$ then a very slight modification of the proof of Theorem 10 gives the necessary conclusion. So we must prove that d_m is a q-moment sequence with weight function in the class F^* if the q-Hausdorff matrix is regular. Suppose first that

$$\sum_{s=0}^{p} |\lambda_{ps}| < K, p = 0, 1, \ldots \tag{71}$$

We rewrite (69) in the form

$$d_n = \frac{1}{(q^{p-n+1};q)_n} \sum_{k=0}^{p-n} [q^k - q^{p-n+1}]_q^n \lambda_{p,p-k} \tag{72}$$

We may write the right side of (72) as a Riemann-Stieltjes integral in the form

$$d_n = \frac{1}{(q^{p-n+1};q)_n} \int_0^1 [t - q^{p-n+1}]_q^n d\Psi_p(t) \tag{73}$$

The weight function $\Psi_p(t)$ is defined by

$$\Psi_p(t) = \begin{cases} 0 & \text{if } t < q^p \\ \lambda_{p0} + \lambda_{p1} & \text{if } q^{p-1} \le t < q^{p-2} \\ \cdots & \\ \lambda_{p0} + \ldots + \lambda_{p,p-1} & \text{if } q \le t < 1 \\ \lambda_{p0} + \ldots \lambda_{pp} & \text{if } 1 \le t \end{cases} \tag{74}$$

The function $\Psi_p(t)$ thus defined is of uniformly bounded variation because $\sum_{s=0}^{p} |\lambda_{ps}| < K, p = 0, 1, \ldots$ We may apply the reasoning that led to equation (62) and then appeal to the Helly-Bray Theorem [9] to conclude that

$$d_n = \int_0^1 t^n d\Psi(t) \tag{75}$$

where $\Psi(t) \in F^*$. Now suppose that $\lim_{p\to\infty} \sum_{s=0}^{p} \lambda_{ps} = 1$. Using (43) we have that

$$\sum_{s=0}^{p} \lambda_{ps} = \int_0^1 d\Lambda(t). \tag{76}$$

We thus have that $\Lambda(1) - \Lambda(0^+) = 1$. Lastly suppose that $\lim_{p\to\infty} \lambda_{ps} = 0$. Then

$$\lim_{p\to\infty} (-1)^s \begin{bmatrix} p \\ s \end{bmatrix} q^{(s^2-s-p^2+p)/2} \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi(t) = 0. \tag{77}$$

The above implies that $\lim_{p\to\infty} \int_0^{q^{p-s}} t^s [t-1]_q^{p-s} d\Psi(t) = 0$. It is not difficult to show that this implies $\Psi(0^+) = \Psi(0) = 0$. ■

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Iterative Algorithms for Multi-Valued Variational Inclusions in Banach Spaces

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In this paper, using fixed point and implicit resolvent equation techniques, we develop some iterative algorithms for a class of variational inclusions involving multi-valued mappings in real Banach space. Further we prove existence of solutions for this class of variational inclusions. Moreover, we discuss convergence criteria for the iterative sequences generated by the iterative algorithms. The theorems presented in this paper, improve, unify and generalize the results of Noor [14-20, and the references therein].

KEY WORDS: Multi-valued variational inclusion, iterative algorithm, implicit resolvent equation, strongly η-accretive mapping, m-accretive mapping, proximal point mapping.

1. INTRODUCTION

Variational inequality theory has emerged as a powerful tool for a wide class of unrelated problems arising in various branches of physical, engineering, pure and applied sciences in a unified and general framework. Variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas; both for their own sake and for their applications. An important and useful generalization of various classes of variational (-like) and quasi-variational (-like) inequali-

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ties is variational (-like) and quasi-variational (-like) inclusions.

In recent years, much attention has been given to develop efficient and implementable numerical methods including projection method and its variant forms, linear approximation, auxiliary principle method, descent and Newton's methods. In 1994, Hassouni and Moudafi [7] introduced and studied a class of variational inclusions and developed a perturbed iterative algorithm for the variational inclusions. Since then Adly [1], Haung [8], Kazmi [10], Ding [2] and Noor [19] obtained some important extensions of the result [7].

Note that most of results established in this direction by a number of authors, see for example, Noor [14-20, and the references therein] are obtained in *Hilbert spaces*.

Very recently, He [21] has shown that if a multi-valued self mapping defined on a Banach space is lower semicontinuous and ϕ-strongly accretive then the value of this mapping at any point of its domain is a singleton set.

In view of above result of He [21], the conditions on multivalued mappings used in establishing the results for the existence of solution and the convergence criteria of the iterative algorithms for multi-valued variational inclusions, see for example Noor [14-20, and relevent references cited therein], made them, in reality, for

single-valued variational inclusions inspite of involving multi-valued mappings. Therefore, methods used previously by many authors, see for example [19], to study the existence of solution and the convergence criteria of the iterative algorithms for multi-valued monotone variational inclusions (inequalities) need improvement.

In this paper, we consider a class of multi-valued variational inclusions in *real Banach spaces* **and show its equivalence with a class of implicit resolvent equations. Using these equivalences, we propose and analyze some iterative algorithms for this class of inclusions. Further we prove the existence of solution and discuss the convergence criteria of the iterative algorithms for the class of multi-valued variational inclusions. The theorems presented in this paper generalize, improve and unify the results given in [19]. The methods developed in this paper can be used to improve and unify the results in [14-19, and the relevent references cited therein].**

2. PRELIMINARIES

Throughout this paper, we assume that E **is a real Banach space equipped with norm** $\|\cdot\|$**;** E^* **is the topological dual space of** E **equipped with norm** $|||\cdot|||$**;** $CB(E)$ **is the family of all nonempty closed and bounded subsets of** E**;** 2^E **is a power set of** E**;** $H(\cdot,\cdot)$ **is the Hausdorff metric on** $CB(E)$ **defined by**

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\},\ \ A,B \in CB(E),$$

$\langle\cdot,\cdot\rangle$ **is the dual pair between** E **and** E^***,** $J : E \longrightarrow 2^{E^*}$ **is the** *normalized*

duality mapping **defined by**

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|x\| = |||f|||\}, \quad x \in E,$$

and j **is the** *selection* **of normalized duality mapping** J**.**
We observe immediately that if $E \equiv H$**, a Hilbert space, then** J **is the identity map on** H**.**

First, we recall and introduce the following definitions.

Definition 2.1. A single-valued mapping $G : E \to E$ **is said to be** γ*-strongly accretive* **if,** $\forall u, v \in E$**,** $\exists\ j(u-v) \in J(u-v)$ **and** $\gamma > 0$ **such that**

$$\langle Gu - Gv, j(u-v)\rangle \geq \gamma\|u-v\|^2.$$

Definition 2.2. A multi-valued mapping $A : E \to 2^E$ **is said to be**

(i) *accretive* **if,** $\forall u, v \in E$**,** $\exists\ j(u-v) \in J(u-v)$ **such that**

$$\langle x-y, j(u-v)\rangle \geq 0, \quad \forall x \in Au,\ y \in Av;$$

(ii) m*-accretive* **if** A **is accretive and** $(I + \rho A)(E) = E$ **for any** $\rho > 0$**, where** I **stands for identity mapping;**

(iii) δ**-**H*-Lipschitz continuous* **if** $\exists \delta > 0$ **such that**

$$H(Au, Av) \leq \delta\|u-v\|, \quad \forall u, v \in E.$$

Definition 2.3. Let $\eta : E \times E \to E$ **and** $A : E \to 2^E$**. A mapping** $N : E \times E \to E$ **is said to be**

(i) *α-strongly η-accretive* **with respect to** A **in the first argument if,** $\forall u, v \in E$, $\exists\ j\eta(u,v) \in J\eta(u,v)$ **and** $\alpha > 0$ **such that**

$$\langle N(x,\cdot) - N(y,\cdot), j\eta(u,v)\rangle \geq \alpha\|u-v\|^2, \quad \forall x \in Au,\ y \in Av;$$

(ii) *β-Lipschitz continuous* **in the first argument if** $\exists \beta > 0$ **such that**

$$\|N(u,\cdot) - N(v,\cdot)\| \leq \beta\|u-v\|, \quad \forall u, v \in E.$$

Remark 2.1. In Definition 2.3, if $\eta(u,v) = u - v, \quad \forall u, v \in E$**, we recover the usual definitions of accretiveness.**

We need the following lemmas in the sequel.

Lemma 2.1 [13]. Let E **be a real Banach space and** $J : E \to 2^{E^*}$ **be the normalized duality mapping. Then for any** $x, y \in E$**,**

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2.2 [21]. Let E **be a real Banach space and** $S : E \to 2^E \setminus \emptyset$ **be a lower semicontinuous and** ϕ**-strongly accretive mapping, then for any** $u \in E$**, Su is a single point.**

Lemma 2.3 [9]. Let E **be a real Banach space and** $A : D(A) \subseteq E \to 2^E$ **be an** m**-accretive mapping. Then the mapping** $J_\rho^A : E \to D(A)$ **associated with** A **defined by** $J_\rho^A(u) = (I + \rho A)^{-1}(u), \quad u \in E$**, for any** $\rho > 0$**, is single valued and nonexpansive.**

Note that $J_\rho^A(u)$ **is so called resolvent (or proximal) mapping.**

Let $N : E \times E \to E$ and $G : E \to E$ be two single-valued mappings; let $S, T, A : E \to CB(E)$ be three multi-valued mappings and $M : E \times E \to 2^E$ be a multi-valued mapping such that for each $u \in E,\ M(\cdot, u)$ is m-accretive. We consider the following multi-valued variational inclusion problem (in short, MVIP):

Find $u \in E,\ x \in Su,\ y \in Tu$ and $z \in Au$ such that $Gu \in E$ and

$$0 \in N(x, y) + M(Gu, z). \tag{2.1}$$

Special Cases of MVIP (2.1)

I. If $E \equiv H$, a real Hilbert space, and if A is identity mapping then MVIP (2.1) reduces to the problem studied by Noor [19].

II. If $E \equiv H$, a real Hilbert space, A is identity mapping, $M(\cdot, u) = \partial\phi(\cdot, u)$, where $\phi : H \times H \to \mathbb{R} \cup \{+\infty\}$ is such that $\phi(\cdot, u)$ is a proper and lower semi-continuous functional for all $u \in H$, and $\partial\phi(\cdot, u)$ denotes the subdifferential of $\phi(\cdot, u)$, then MVIP (2.1) reduces to variational inequality problem of finding $u \in H, x \in Su$ and $y \in Tu$, such that

$$\langle N(x, y), v - Gu\rangle \geq \phi(Gu, u) - \phi(v, u), \quad \forall v \in H,$$

which is similar to the problem considered by Ding [3].

We remark that for suitable choices of N, M, S, T and G, MVIP(2.1) reduces to various classes of variational inclusions and variational inequalities, see for example [1-3,7-10,14-20], studied by many authors in the setting of Hilbert spaces. Our problem MVIP(2.1) is

also set in more general real Banach space.

Let $R_\rho^{M(\cdot,z)} = I - J_\rho^{M(\cdot,z)}$, where I is the identity mapping on E and $J_\rho^{M(\cdot,z)} = (I + \rho M(\cdot, z))^{-1}$ is a resolvent mapping for all $z \in E$ and $\rho > 0$, a constant.

Let $N : E \times E \to E$ and $G : E \to E$ be two single-valued mappings; $S, T, A : E \to CB(E)$ be three multi-valued mappings and $M : E \times E \to 2^E$ be a multi-valued mapping such that for each $u \in E,\ M(\cdot, u)$ is m-accretive. We consider the following problem of finding $w, u \in E, x \in Su, y \in Tu$ and $z \in Au$ such that $Gu \in E$ and

$$N(x, y) + \rho^{-1} R_\rho^{M(\cdot,z)} w = 0. \tag{2.2}$$

Equation (2.2) is called the implicit resolvent equation, which includes as special cases, many known resolvent equations and Wiener-Hopf equations, see for example [14,15,17,19,23] and the references therein.

3. ITERATIVE ALGORITHMS

The following lemma which will be used in the sequel, is an immediate consequence of the definition of $J_\rho^{M(\cdot,z)}$.

Lemma 3.1. **(u, x, y, z), where $u \in E, x \in Su,\ y \in Tu$ and $z \in Au$, is a solution of MVIP (2.1) if and only if it satisfies the relation**

$$Gu = J_\rho^{M(\cdot,z)}(Gu - \rho N(x, y)), \tag{3.1}$$

where $J_\rho^{M(\cdot,z)} = (I + \rho M(\cdot, z))^{-1}$ and $\rho > 0$ is a constant.

Using Lemma 3.1 and Nadler's technique [12], we develop an iterative algorithm for finding the approximate solution of MVIP (2.1) as follows.

Iterative Algorithm 3.1. Let $N : E \times E \to E$, $G : E \to E$ and $S, T, A : E \to CB(E)$ be such that for each $u \in E$, $Q(u) \subseteq G(E)$, where $Q : E \to 2^E$ is a multi-valued mapping defined by

$$Q(u) = \bigcup_{x \in Su} \bigcup_{y \in Tu} \bigcup_{z \in Au} \left(J_\rho^{M(\cdot,z)}(Gu - \rho N(x, y)) \right), \tag{3.2}$$

where $M : E \times E \to 2^E$ is a multi-valued mapping such that for each $u \in E$, $M(\cdot, u)$ is m-accretive.

For given $u_0 \in E, x_0 \in Su_0, y_0 \in Tu_0$, and $z_0 \in Au_0$, and let

$$w_0 = (1 - \lambda)Gu_0 + \lambda J_\rho^{M(\cdot,z_0)}(Gu_0 - \rho N(x_0, y_0)) \in Q(u_0) \subseteq G(E).$$

Hence, there exists $u_1 \in E$ such that $w_0 = Gu_1$. Since $x_0 \in Su_0 \in CB(E), y_0 \in Tu_0 \in CB(E)$ and $z_0 \in Au_0 \in CB(E)$ then by Nadler's theorem [12], there exist $x_1 \in Su_1, y_1 \in Tu_1$ and $z_1 \in Au_1$ such that

$$\|x_1 - x_0\| \leq (1 + (1 + 0)^{-1})H(Su_1, Su_0),$$

$$\|y_1 - y_0\| \leq (1 + (1 + 0)^{-1})H(Tu_1, Tu_0),$$

$$\|z_1 - z_0\| \leq (1 + (1 + 0)^{-1})H(Au_1, Au_0).$$

Let

$$w_1 = (1 - \lambda)Gu_1 + \lambda J_\rho^{M(\cdot,z_1)}(Gu_1 - \rho N(x_1, y_1)) \in Q(u_1) \subseteq G(E).$$

Hence, there exists $u_2 \in E$ such that $w_1 = Gu_2$. By induction, we can define iterative sequences $\{u_n\},\ \{Gu_n\},\ \{x_n\},\ \{y_n\}$ and $\{z_n\}$ as follows:

$$Gu_{n+1} = (1-\lambda)Gu_n + \lambda J_\rho^{M(.,z_n)}(Gu_n - \rho N(x_n, y_n)), \tag{3.3}$$

$$x_n \in Su_n \ : \ \|x_{n+1} - x_n\| \leq (1 + (1+n)^{-1})H(Su_{n+1}, Su_n), \tag{3.4}$$

$$y_n \in Tu_n \ : \ \|y_{n+1} - y_n\| \leq (1 + (1+n)^{-1})H(Tu_{n+1}, Tu_n), \tag{3.5}$$

$$z_n \in Au_n \ : \ \|z_{n+1} - z_n\| \leq (1 + (1+n)^{-1})H(Au_{n+1}, Au_n), \tag{3.4}$$

where $n = 0, 1, 2, 3, ...$ and $\rho > 0$ is a constant and $0 < \lambda \leq 1$ is a relaxation parameter.

Next lemma shows the equivalence between MVIP (2.1) and implicit resolvent equation(2.2).

Lemma 3.2. **(u, x, y, z), where $u \in E, x \in Su,\ y \in Tu$ and $z \in Au$, is a solution of MVIP (2.1) if and only if $(w, u, x, y, z),\ w \in E$, is a solution of implicit resolvent equation(2.3), where**

$$Gu = J_\rho^{M(\cdot,z)} w, \tag{3.7}$$

$$w = Gu - \rho N(x, y), \tag{3.8}$$

and $\rho > 0$ is a constant.

The proof follows the same lines of proof of Theorem 4.1 [19] and hence is omitted.

Now, the implicit resolvent equation (2.2) can be written as

$$R_{\rho}^{M(\cdot,z)}w = -\rho N(x,y)$$

which implies

$$\begin{aligned} w &= J_{\rho}^{M(\cdot,z)}w - \rho N(x,y) \\ &= Gu - \rho N(x,y). \end{aligned}$$

This fixed point formulation and Nadler's technique [12] allows us to suggest the following iterative algorithm.

Iterative Algorithm 3.2. For given $w_0, u_0 \in E, x_0 \in Su_0, y_0 \in Tu_0$ **and** $z_0 \in Au_0$**, define iterative sequences** $\{w_n\},\ \{u_n\},\ \{x_n\},\ \{y_n\}$ **and** $\{z_n\}$ **as follows:**

$$Gu_n = J_{\rho}^{M(.,z_n)}w_n, \tag{3.9}$$

$$x_n \in Su_n \ : \ \|x_{n+1} - x_n\| \leq (1 + (1+n)^{-1})H(Su_{n+1}, Su_n),$$

$$y_n \in Tu_n \ : \ \|y_{n+1} - y_n\| \leq (1 + (1+n)^{-1})H(Tu_{n+1}, Tu_n),$$

$$z_n \in Au_n \ : \ \|z_{n+1} - z_n\| \leq (1 + (1+n)^{-1})H(Au_{n+1}, Au_n),$$

$$w_{n+1} = (1-\lambda)w_n + \lambda[Gu_n - \rho N(x_n, y_n)], \tag{3.10}$$

where $n = 0, 1, 2, 3, ...;\ \rho > 0$ **is a constant and** $0 < \lambda < 1$ **is a relaxation parameter.**

We remark that Iterative Algorithms 3.1 and 3.2 include as special cases many known iterative algorithms in Hilbert spaces, see [19] and the references therein. Moreover, one can also develop the iterative algorithms similar to Algorithms 4.2 and 4.3 of Noor

[19] for MVIP (2.1) in Banach space.

In the next section, we prove the existence of solution of MVIP (2.1) and discuss the convergence criteria for the iterative sequences generated by Iterative Algorithms 3.1 and 3.2.

4. EXISTENCE OF SOLUTION AND CONVERGENCE CRITERIA

Theorem 4.1. **Let E be a real Banach space and $\eta : E\times E \to E$ be τ-Lipschitz continuous. Let $S, T, A : E \to CB(E)$ and $G : E \to E$ be σ-H-Lipschitz continuous, δ-H-Lipschitz continuous, ξ-H-Lipschitz continuous and ϵ-Lipschitz continuous mappings, respectively, and $(G - I) : E \to E$ be ν-strongly accretive mapping, where I is the identity mapping on E. Let $N : E \times E \to E$ be β-Lipschitz continuous and γ-Lipschitz continuous with first and second arguments, respectively, and be α-strongly η-accretive with respect to S in the first argument. Let $M : E\times E \to 2^E$ be such that for each fixed $z \in E$, $M(., z)$ is m-accretive mapping and let for each $u \in E,\ \ Q(u) \subseteq G(E)$, where Q is defined by (3.2). Suppose that there exist $\rho > 0$ and $l > 0$ such that for each $z_1, z_2, v \in E$,**

$$\|J_\rho^{M(.,z_1)}(v) - J_\rho^{M(.,z_2)}(v)\| \leq l\|z_1 - z_2\|, \tag{4.1}$$

and

$$\left|\rho\lambda - \frac{\lambda(\alpha - \sigma\beta k_2) - \gamma\delta k_3}{2\sigma^2\beta^2 - \gamma^2\delta^2}\right|$$

$$< \frac{\sqrt{[\lambda(\alpha-\sigma\beta k_2)-\gamma\delta k_3]^2+(2\sigma^2\beta^2-\gamma^2\delta^2)(k_3^2-\lambda^2\epsilon^2)}}{2\sigma^2\beta^2-\gamma^2\delta^2} \tag{4.2}$$

$$[\lambda(\alpha-\beta\sigma k_2)-\gamma\delta k_3]^2+(2\sigma^2\beta^2-\gamma^2\delta^2)(k_3^2-\lambda^2\epsilon^2)>0;\ \sqrt{2}\beta\sigma>\gamma\delta$$

where $k_1=\sqrt{2\nu+1};\ k_2=\epsilon+\tau$ **and** $k_3=k_1-(1-\lambda)\epsilon-\lambda l\xi$**.**
Then the iterative sequences $\{u_n\},\{x_n\},\{y_n\}$ **and** $\{z_n\}$ **generated by Iterative Algorithm 3.1 converge strongly to** u^*,x^*,y^* **and** z^***, respectively, and** (u^*,x^*,y^*,z^*) **is a solution of MVIP (2.1).**

Proof. Using Lemma 2.1 and ν**-strongly accretiveness of** $G-I$**, we have**

$$\begin{aligned}
&\|u_{n+2}-u_{n+1}\|^2\\
&= \|Gu_{n+2}-Gu_{n+1}+u_{n+2}-u_{n+1}-(Gu_{n+2}-Gu_{n+1})\|^2\\
&\leq \|Gu_{n+2}-Gu_{n+1}\|^2-2\langle(G-I)u_{n+2}-(G-I)u_{n+1},j(u_{n+2}-u_{n+1})\rangle\\
&\leq \|Gu_{n+2}-Gu_{n+1}\|^2-2\nu\|u_{n+2}-u_{n+1}\|^2,
\end{aligned}$$

which implies

$$\|u_{n+2}-u_{n+1}\|\leq\frac{1}{\sqrt{2\nu+1}}\|Gu_{n+2}-Gu_{n+1}\|. \tag{4.3}$$

Next, from Lemma 2.3, (3.3) and (4.1), we have

$$||Gu_{n+2}-Gu_{n+1}||$$

$$
\begin{aligned}
&= (1-\lambda)||Gu_{n+1} - Gu_n|| \\
&\quad +\lambda||J_\rho^{M(.,z_{n+1})}\Big(Gu_{n+1} - \rho N(x_{n+1}, y_{n+1})\Big) - J_\rho^{M(.,u_n)}\Big(Gu_n - \rho N(x_n, y_n)\Big)|| \\
&\leq (1-\lambda)||Gu_{n+1} - Gu_n|| \\
&\quad +\lambda||J_\rho^{M(.,z_{n+1})}\Big(Gu_n - \rho N(x_n, y_n)\Big) - J_\rho^{M(.,z_n)}\Big(Gu_n - \rho N(x_n, y_n)\Big)|| \\
&\quad +\lambda||J_\rho^{M(.,z_{n+1})}\Big(Gu_{n+1} - \rho N(x_{n+1}, y_{n+1})\Big) - J_\rho^{M(.,z_{n+1})}\Big(Gu_n - \rho N(x_n, y_n)\Big)|| \\
&\leq (1-\lambda)||Gu_{n+1} - Gu_n|| + \lambda l||z_{n+1} - z_n|| \\
&\quad +\lambda||Gu_{n+1} - Gu_n - \rho\Big[N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1})\Big]|| \\
&\quad +\lambda\rho||N(x_n, y_{n+1}) - N(x_n, y_n)||.
\end{aligned} \tag{4.4}
$$

Since G is ϵ-Lipschitz continuous mapping, we have

$$||Gu_{n+1} - Gu_n|| \leq \epsilon||u_{n+1} - u_n||. \tag{4.5}$$

Since A and T are ξ-H-Lipschitz continuous and δ-H-Lipschitz continuous, respectively, and N is γ-Lipschitz continuous, in the second argument, we have

$$\|z_{n+1} - z_n\| \leq (1 + (1+n)^{-1})H(Au_{n+1}, Au_n) \leq \xi(1 + (1+n)^{-1})\|u_{n+1} - u_n\| \tag{4.6}$$

and

$$
\begin{aligned}
||N(x_n, y_{n+1}) - N(x_n, y_n)|| &\leq \gamma||y_{n+1} - y_n|| \\
&\leq \gamma(1 + (1+n)^{-1})H(Tu_{n+1}, Tu_n), \\
&\leq \gamma\delta(1 + (1+n)^{-1})\|u_{n+1} - u_n\|.
\end{aligned} \tag{4.7}
$$

Furthermore, since N is α-H-strongly η-accretive with respect

to S and β-Lipschitz continuous in the first argument and S is σ-H-Lipschitz continuous, by using Lemma 2.1, we obtain that

$$
\begin{aligned}
&||Gu_{n+1}-Gu_n-\rho[N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})]||^2\\
&\leq ||Gu_{n+1}-Gu_n||^2-2\rho\langle N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}),j(Gu_{n+1}-Gu_n\\
&\quad -\rho[N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})])\rangle\\
&\leq \epsilon^2||u_{n+1}-u_n||^2-2\rho\langle N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}),j(\eta(u_{n+1},u_n))\rangle\\
&\quad -2\rho\langle N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}),j(Gu_{n+1}-Gu_n-\rho[N(x_{n+1},y_{n+1})\\
&\quad -N(x_n,y_{n+1})])-j(\eta(u_{n+1},u_n))\rangle\\
&\leq \epsilon^2||u_{n+1}-u_n||^2-2\rho\alpha||u_{n+1}-u_n||^2+2\rho||N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})||\\
&\quad \times\Big[||Gu_{n+1}-Gu_n||+\rho||N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})||+||\eta(u_{n+1},u_n)||\Big]\\
&\leq (\epsilon^2-2\rho\alpha)||u_{n+1}-u_n||^2+2\rho\beta||x_{n+1}-x_n||\Big[(\epsilon+\tau)||u_{n+1}-u_n||\\
&\quad +\rho\beta||x_{n+1}-x_n||\Big]\\
&= \Big[\epsilon^2-2\rho\alpha+2\rho\beta\sigma(\epsilon+\tau)(1+(1+n)^{-1})+2\rho^2\beta^2\sigma^2(1+(1+n)^{-1})^2\Big]\\
&\quad \times||u_{n+1}-u_n||^2. \qquad (4.8)
\end{aligned}
$$

Combining (4.3)-(4.8), we have

$$||u_{n+2}-u_{n+1}||\leq\theta_n||u_{n+1}-u_n||, \qquad (4.9)$$

where

$$\theta_n:=\frac{1}{\sqrt{2\nu+1}}\Big[(1-\lambda)\epsilon+\lambda l\xi(1+(1+n)^{-1})+\lambda\rho\gamma\delta(1+(1+n)^{-1})$$

$$+\lambda\sqrt{\epsilon^2 - 2\rho\alpha + 2\rho\beta\sigma(\epsilon+\tau)(1+(1+n)^{-1}) + 2\rho^2\beta^2\sigma^2(1+(1+n)^{-1})^2}\Big]. \tag{4.10}$$

Letting $n \to \infty$, we obtain that $\theta_n \to \theta$, where

$$\theta := \frac{1}{\sqrt{2\nu+1}}\Big[(1-\lambda)\epsilon + \lambda l\xi + \lambda\rho\gamma\delta + \lambda\sqrt{\epsilon^2 - 2\rho\alpha + 2\rho\beta\sigma(\epsilon+\tau) + 2\rho^2\beta^2\sigma^2}\ \Big]. \tag{4.11}$$

Since $0 < \theta < 1$ by condition (4.2). Hence $\theta_n < 1$ for n sufficiently large. Therefore (4.9) implies that $\{u_n\}$ is a Cauchy sequence in E, and hence there exists $u^* \in E$ such that $u_n \to u^*$ as $n \to \infty$. By the H-Lipschitz continuity of S and (3.4), we have

$$||x_{n+1} - x_n|| \leq (1+(1+n)^{-1})H(Su_{n+1}, Su_n) \leq \sigma(1+(1+n)^{-1})||u_{n+1} - u_n||.$$

It follows that $\{x_n\}$ is also a Cauchy sequence. Similarly, we can show that $\{y_n\}, \{z_n\}$ and $\{Gu_n\}$ are also Cauchy sequences in E. Hence there exist $x^*, y^*, z^* \in E$ such that $Gu_n \to Gu^*$, $x_n \to x^*$, $y_n \to y^*$ and $z_n \to z^*$ as $n \to \infty$. Furthermore, since $x_n \in Su_n$, we have

$$\begin{aligned} d(x^*, Su^*) &\leq ||x^* - x_n|| + d(x_n, Su^*) \\ &\leq ||x^* - x_n|| + H(Su_n, Su^*) \\ &\leq ||x^* - x_n|| + \sigma||u_n - u^*|| \to 0, \end{aligned}$$

and hence $x^* \in Su^*$. Similarly, $y^* \in Tu^*,\ z^* \in Au^*$.

From Iterative Algorithm 3.1 and continuity of mappings G, N, S, T, A and $J_\rho^{M(\cdot,z^*)}$ and condition (4.1), it follows that

$$Gu^* = J_\rho^{M(\cdot,z^*)}\left(Gu^* - \rho N(x^*, y^*)\right).$$

Thus, by Lemma 3.1, it follow that (u^*, x^*, y^*, z^*) is a solution of MVIP (2.1) and this completes the proof.

Remark 4.1. It is clear that $\nu \leq \epsilon$; $\alpha \leq \beta\sigma\tau$. Further condition (4.2) is true for suitable values of constants, for example, $\alpha = \beta = \sigma = \gamma = \tau = \xi = \delta = 1$; $\nu = \epsilon = 0.5$; $l = 0.1$; $\rho \in (0, 0.3)$ and $\lambda \in (0, 1]$.

Theorem 4.2. Let E be a real Banach space and the mappings $\eta, S, T, A, G, N, G-I$ and M be the same as in Theorem 4.1. Assume that conditions (4.1) and (4.2) with $\lambda = 1$ of Theorem 4.1 hold. Then the iterative sequences $\{w_n\}, \{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by Iterative Algorithm 3.2 converge strongly to w^*, u^*, x^*, y^* and z^*, respectively, and $(w^*, u^*, x^*, y^*, z^*)$ is a solution of implicit resolvent equation (2.2).

Proof. From Iterative Algorithm 3.2 and using (4.7) and (4.8), we have

$$\|w_{n+2}-w_{n+1}\| \leq (1-\lambda)\|w_{n+1}-w_n\|+\lambda||Gu_{n+1}-Gu_n-\rho[N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})]||$$

$$+ \lambda\rho\|N(x_n, y_{n+1}) + N(x_n, y_n)\|$$

$$\leq (1-\lambda)\|w_{n+1} - w_n\| + \lambda\theta'_n\|u_{n+1} - u_n\|, \tag{4.12}$$

where

$$\theta'_n := \rho\gamma\delta(1 + (1+n)^{-1})$$

$$+ \sqrt{\epsilon^2 - 2\rho\alpha + 2\rho\beta\sigma(\epsilon+\tau)(1+(1+n)^{-1}) + 2\rho^2\beta^2\sigma^2(1+(1+n)^{-1})^2}.$$

From (3.9), (4.1), (4.3), and (4.6), we have

$$\|u_{n+2} - u_{n+1}\| \leq \frac{1}{\sqrt{2\nu+1}}\|Gu_{n+2} - Gu_{n+1}\|$$

$$\leq \frac{1}{\sqrt{2\nu+1}}\left[l\xi(1+(1+n)^{-1})\|u_{n+1}-u_n\|+\|w_{n+1}-w_n\|\right]$$

$$\|u_{n+2}-u_{n+1}\| \leq \frac{1}{\sqrt{2\nu+1}-l\xi(1+(1+n)^{-1})}\|w_{n+1}-w_n\|. \tag{4.13}$$

Combining (4.12)and (4.13), we have

$$\|w_{n+2}-w_{n+1}\| \leq [1-\lambda(1-\theta_n)]\|w_{n+1}-w_n\|, \tag{4.14}$$

where

$$\theta_n = \theta'_n\left[\frac{1}{\sqrt{2\nu+1}-l\xi(1+(1+n)^{-1})}\right].$$

Letting $n \to \infty, \quad \theta_n \to \theta$**, where**

$$\theta = \frac{\rho\gamma\delta+\sqrt{\epsilon^2-2\rho\alpha+2\rho\beta\sigma(\epsilon+\tau)+2\rho^2\beta^2\sigma^2}}{\sqrt{2\nu+1}-l\xi}.$$

Since $0<\theta<1$ **by condition (4.2) with** $\lambda=1$**. Hence** $\theta_n<1$ **for** n **sufficiently large. Therefore, (4.14) implies that** $\{w_n\}$ **is a Cauchy sequence in** E**, and hence there exists** $w^* \in E$ **such that** $w_n \to w^*$ **as** $n \to \infty$**. From (4.13) and from Theorem 4.1, we see that sequences** $\{u_n\}$**,** $\{x_n\}$**,** $\{y_n\}$ **and** $\{z_n\}$ **are Cauchy sequences in** E**. Hence there exist** $u^*,\ x^*,\ y^*$ **and** z^* **in** E **such that** $u_n \to u^*$**,** $x_n \to x^*$**,** $y_n \to y^*$ **and** $z_n \to z^*$ **as** $n \to \infty$**. Using the technique of Theorem 4.1, we see that** $x^* \in Su^*,\ y^* \in Tu^*,\ z^* \in Au^*$ **and thus, the continuity of mappings** S, T, A, M, G **and** $J_\rho^{M(\cdot,z)}$ **and Iterative Algorithm 3.2 give that**

$$w^* = Gu^* - \rho N(x^*, y^*) = J_\rho^{M(\cdot,z^*)}w^* - \rho N(x^*, y^*) \in E.$$

Hence, by Lemma 3.2, it follows that $w^*, u^* \in E$**,** $x^* \in Su^*,\ y^* \in Tu^*,\ z^* \in Au^*$ **is a solution of the implicit resolvent equation (2.2). This completes the proof.**

Remark 4.2. If we take $\eta(u,v) = u - v,\ \forall u, v \in E$ in Theorems 4.1 and 4.2, then by Lemma 2.2, the multi-valued mapping $C : E \to 2^E \setminus \emptyset$ defined by $Cu = N(Su, y)$, for fixed $y \in E$ and for all $u \in E$, is a single-valued mapping.

In this case, using the technique of He [21], we estimate $\|N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1})\|$ as follows:

For any $x_{n+1}, x'_{n+1} \in Su_{n+1}, \quad N(x_{n+1}, y_{n+1}) = N(x'_{n+1}, y_{n+1})$. On the other hand, for $x_n \in Su_n \in CB(E)$, there exists a $x'_{n+1} \in Su_{n+1}$ such that

$$\|x'_{n+1} - x_n\| \leq (1 + (1+n)^{-1}) H(Su_{n+1}, Su_n).$$

Hence we have

$$\begin{aligned} \|N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1})\| &= \|N(x'_{n+1}, y_{n+1}) - N(x_n, y_{n+1})\| \\ &\leq \beta \|x'_{n+1} - x_n\| \\ &\leq \beta(1 + (1+n)^{-1}) H(Su_{n+1}, Su_n) \\ &\leq \beta\sigma(1 + (1+n)^{-1}) \|u_{n+1} - u_n\|. \end{aligned} \tag{4.15}$$

Also (4.7) holds if and only if $N(S(\cdot), T(\cdot))$ is single-valued. Indeed, if $N(S(\cdot), T(\cdot))$ is single-valued, then (4.7) can be proved as (4.15). Conversely, from (4.7) and (4.15), we have

$$\begin{aligned} \|N(x_{n+1}, y_{n+1}) - N(x_n, y_n)\| &= \|N(x_{n+1}, y_{n+1}) - N(x_n, y_{n+1})\| \\ &\quad + \|N(x_n, y_{n+1}) - N(x_n, y_n)\| \\ &\leq (\beta\sigma + \gamma + \delta)(1 + (1+n)^{-1}) \|u_{n+1} - u_n\|. \end{aligned}$$

For any $u \in E$, let $u_{n+1} = u_n = u$ in preceding inequality, then for any $(x_{n+1}, y_{n+1}), (x_n, y_n) \in Su \times Tu$, it follow that

$$\|N(x_{n+1}, y_{n+1}) - N(x_n, y_n)\| = 0.$$

Thus $N(x_{n+1}, y_{n+1}) = N(x_n, y_n)$, which implies that $N(S(\cdot), T(\cdot))$ is single-valued. Since inequality (4.7) has been used in the proof of Theorems, it should be regarded as an additional condition.

Further, in view of the single-valuedness of operator $N(S(\cdot), T(\cdot))$, we can release $x_n,\ y_n$ from the restrictions that

$$x_n \in Su_n \ : \ \|x_{n+1} - x_n\| \leq (1 + (1+n)^{-1})H(Su_{n+1}, Su_n),$$

$$y_n \in Tu_n \ : \ \|y_{n+1} - y_n\| \leq (1 + (1+n)^{-1})H(Tu_{n+1}, Tu_n).$$

Consequently the limits of $\{x_n\}$ and $\{y_n\}$ need not be considered. When $\lim_{n\to\infty} u_n = u^*, \quad \lim_{n\to\infty} w_n = w^*$ and $\lim_{n\to\infty} z_n = z^*$ are obtained then for each $x \in Su^*$ and each $y \in Tu^*, \quad (u^*, x, y, z^*)$ and (u^*, w^*, x, y, z^*) are solutions of MVIP(2.1) and implicit resolvent equation (2.2) respectively.

Remark 4.3. In view of Remark 4.2, Theorems 4.1 and 4.2 for variational inclusion (2.1) considered by Noor [19] in reality, are for single- valued variational inclusion inspite of involving multi-valued mappings.

Remark 4.4. Our Theorems 4.1 and 4.2 generalize, improve and unify the results given in Noor [19] and the references therein.

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Pointwise Weight Approximation by Left Gamma Quasi–Interpolants *

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Abstract. Recently some classical operator quasi-interploants were introduced to obtain much faster convergence. Müller [8] gave Gamma left quasi-interpolants and obtained approximation equivalence theorem with $\omega_{\varphi}^{2r}(f,t)_p$. In this paper we extend above result from two side in L_∞: one is we study weight approximation, the other we use modulus $\omega_{\varphi^\lambda}^{2r}(f,t)_\infty$ which unified classical modulus and Ditzian-Totik modulus.

Key words and phrases: Gamma operator, quasi-interpolants, weight approximation, equivalent theorem, modulus of smoothness.

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1 Introduction

Gamma operator is given by

$$G_n(f,x)=\int_0^\infty g_n(x,t)f\left(\frac{n}{t}\right)dt,\ \ x\in[0,\infty),$$
$$g_n(x,t)=\frac{x^{n+1}}{n!}e^{-xt}t^n. \tag{1.1}$$

The other representation of this operator is

$$G_n(f,x)=\frac{1}{n!}\int_0^\infty e^{-t}t^n f(\frac{nx}{t})dt. \tag{1.2}$$

These operators have been introduced in [6], and investigated in subsequent papers (e.g. [2], [5], [7], [10]).

Ditzian [1] introduced $\omega_{\varphi^\lambda}^2(f,t)$ and gave a direct result for Bernstein operators. He extended the approximation results on $\omega_{\varphi}^2(f,t)$ and $\omega^2(f,t)$. In [3] and [4] we discussed Szasz and Gamma operators by $\omega_{\varphi^\lambda}^{2r}(f,t)$ and obtained same interesting results.

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In [9] so-called left Bernstein quasi-interpolants were introduced. In this way Müller [8] obtained left Gamma quasi-interpolants

$$G_n^{(k)}(f,x)=\sum_{j=0}^{k}\alpha_j^n(x)D^jG_n(f,x),\qquad 0\le k\le n. \tag{1.3}$$

and gave a approximation equivalent theorem: for $f\in L_p[0,\infty)$, $1\le p\le\infty$, $\varphi(x)=x$, $n\ge 4r$, $r\in N$, and $0<\alpha<r$ the following statements are equivalent, that is

$$\|G_n^{(2r-1)}f-f\|_p=O(n^{-\alpha})\Longleftrightarrow\omega_\varphi^{2r}(f,t)_p=O(t^{2\alpha}). \tag{1.4}$$

In this paper we will consider weight approximation for $G_n^{(2r-1)}(f,x)$ in L_∞-spaces with the unified modulus $\omega_{\varphi^\lambda}^{2r}(f,t)_w$, where $w(x)=x^a(1+x)^b\quad(a\ge 0,\ b$ is arbitrary). Our main result is that for $f\in L_\infty[0,\infty)$, $0\le\lambda\le 1$, $\varphi(x)=x$, $w(x)=x^a(1+x)^b$, $n\ge 4r$, and $0<\alpha<2r$, then

$$|w(x)\big(G_n^{(2r-1)}(f,x)-f(x)\big)|=O\Big(\big(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\big)^\alpha\Big)\Longleftrightarrow\omega_{\varphi^\lambda}^{2r}(f,t)_w=O\big(t^\alpha\big). \tag{1.5}$$

In L_∞-spaces, (1.5) extends (1.4), when $\lambda=1$, $a=b=0$ then (1.5) is (1.4).

Throughout this paper $\|\cdot\|$ denotes $\|\cdot\|_\infty$, C denotes a positive constant not necessarily the same at each occurrence.

2 Preliminaries and Lemmas

Suppose $G_n^{(k)}(f,x)=\sum\limits_{j=0}^{k}\alpha_j^n(x)D^jG_n(f,x)$. At first we list some related properties of $G_n^{(k)}(f,x)$, which can be found in [8].

(1) For $j\in N_0$, $n\ge j$, we have $\alpha_j^n(x)\in\Pi_j$ (space of algebraic polynomials of degree at most j) and

$$\alpha_j^n(x)=\big(\frac{x}{n}\big)^jL_j^{(n-j)}(n),\quad \alpha_0^n(x)=1,\quad \alpha_1^n(x)=0, \tag{2.1}$$

where for $\alpha\in R$

$$L_j^{(\alpha)}(x)=\sum_{r=0}^{j}(-1)^r\binom{j+\alpha}{j-r}\frac{x^r}{r!} \tag{2.2}$$

is the Laguerre polynomial of degree j.

(2) For $j\in N_0$ and $n\ge j$

$$\Big|\frac{1}{n^j}L_j^{(n-j)}(n)\Big|\le Cn^{-\frac{j}{2}}. \tag{2.3}$$

(3) If $p\in\Pi_k$, then

$$G_n^{(k)}(p,x)=p(x). \tag{2.4}$$

(4)

$$\frac{\partial^m}{\partial x^m}g_n(x,t)=\frac{m!}{x^m}g_n(x,t)L_m^{(n+1-m)}(xt). \tag{2.5}$$

(5)

$$\begin{aligned}\big(G_nf\big)^{(2r)}(x) &= \frac{n^{2r}}{n!}\int_0^\infty e^{-t}t^{n-2r}f^{(2r)}\big(\frac{nx}{t}\big)dt \\ &= \frac{n^{2r}(n-2r)!}{n!}\int_0^\infty g_{n-2r}(x,u)f^{(2r)}\big(\frac{n}{u}\big)du.\end{aligned} \tag{2.6}$$

(6)

$$\int_0^\infty e^{-t}t^\alpha\big|L_j^{(\alpha)}(t)\big|^2dt = \frac{\Gamma(j+\alpha+1)}{j!} \qquad \text{for } \alpha > -1. \tag{2.7}$$

(7) For $m,n,l \in N_0$,

$$\frac{1}{(n+l)!}\int_0^\infty e^{-t}t^{n+l}\big(\frac{nx}{t}-x\big)^m dt \le C\frac{x^m}{n^{[(m+1)/2]}}. \tag{2.8}$$

Next we give two lemmas.

Lemma 2.1. *(1) Let $w(x) = x^a(1+x)^b$, $a \ge 0$, $b \in R$, $x,u \in (0,\infty)$ then*

$$\frac{w(x)}{w(u)} \le 2^{|b|}\Big(\big(\frac{x}{u}\big)^a + \big(\frac{x}{u}\big)^{a+b}\Big). \tag{2.9}$$

(2) For $\forall\ \beta \in R$ we have

$$\frac{1}{n!}\int_0^\infty e^{-t}t^n\big(\frac{n}{t}\big)^\beta dt \le C(\beta). \tag{2.10}$$

Proof. (1) For $b \ge 0$,

$$\frac{w(x)}{w(u)} \le \big(\frac{x}{u}\big)^a\big(1+\frac{x}{u}\big)^b \le 2^b\Big(\big(\frac{x}{u}\big)^a + \big(\frac{x}{u}\big)^{a+b}\Big).$$

For $b < 0$,

$$\frac{w(x)}{w(u)} \le \big(\frac{x}{u}\big)^a\big(\frac{1+u}{1+x}\big)^{-b} \le \big(\frac{x}{u}\big)^a\big(1+\frac{u}{x}\big)^{-b} \le 2^{|b|}\Big(\big(\frac{x}{u}\big)^a + \big(\frac{x}{u}\big)^{a+b}\Big).$$

(2) By direct computation or [2, p165] we have (2.10) □

Lemma 2.2. *(The boundedness of $G_n^{(k)}$ in weighted norm)*

For $n \ge k$, we have

$$\|wG_n^{(k)}(f)\| \le C\|wf\|. \tag{2.11}$$

Proof.

$$|w(x)G_n^{(k)}(f,x)| \le |w(x)G_n(f,x)| + |w(x)\sum_{j=2}^k \alpha_j^n(x)D^jG_n(f,x)|. \tag{2.12}$$

From [2, p165] we have

$$|w(x)G_n(f,x)| \le C\|wf\|. \tag{2.13}$$

By (1.1), (2.5), (2.7), (2.8) and (2.10) we get

$$
\begin{aligned}
&|w(x)D^jG_n(f,x)|\\
=&\big|w(x)\int_0^\infty \frac{\partial^j}{\partial x^j}g_n(x,t)f\big(\frac{n}{t}\big)dt\big|\\
\le&\big|w(x)\int_0^\infty \frac{j!}{x^j}g_n(x,t)L_j^{(n+1-j)}(xt)w^{-1}\big(\frac{n}{t}\big)dt\big|\ \|wf\|\\
=&\big|w(x)\frac{j!}{n!}\int_0^\infty x^{n+1-j}e^{-tx}t^nL_j^{(n+1-j)}(xt)w^{-1}\big(\frac{n}{t}\big)dt\big|\ \|wf\|\\
\le&C\frac{x^{-j}}{n!}\int_0^\infty e^{-u}u^n\big|L_j^{(n+1-j)}(u)\big|\frac{wx}{w\big(\frac{nx}{u}\big)}du\ \|wf\|\\
\le&Cx^{-j}\Big(\frac{1}{n!}\int_0^\infty e^{-u}u^{n+1-j}\big|L_j^{(n+1-j)}(u)\big|^2du\Big)^{\frac12}\Big(\frac{1}{n!}\int_0^\infty e^{-u}u^{n-1+j}\Big(\big(\frac{u}{n}\big)^a+\big(\frac{u}{n}\big)^{a+b}\Big)^2du\Big)^{\frac12}\ \|wf\|\\
\le&Cx^{-j}\Big(\frac{1}{n!}\frac{(n+1)!}{j!}\Big)^{\frac12}\Big(\frac{(n+j-1)!}{n!}\Big)^{\frac12}\ \|wf\|\\
\le&Cx^{-j}n^{\frac12}n^{\frac{j-1}{2}}\ ||wf\|.
\end{aligned}
\tag{2.14}
$$

Notice that

$$|\alpha_j^n(x)|\le Cn^{-\frac{j}{2}}x^j \tag{2.15}$$

and [2, p161]

$$\|w(x)G_n(f,x)\|\le C\|w(x)f(x)\|. \tag{2.16}$$

From (2.12)–(2.16) we know (2.11) is valid. □

Now we give some definitions of modulus of smoothness and K- functional (cf. [2]).

$$\omega^r_{\varphi^\lambda}(f,t)_w=\begin{cases}\sup\limits_{0<h\le t}\|w\Delta^r_{h\varphi^\lambda}f\|, & a=0,\\ \sup\limits_{0<h\le t}\|w\Delta^r_{\varphi^\lambda}f\|_{[t^*,\infty)}+\sup\limits_{0<h\le t^*}\|w\overrightarrow{\Delta}^r_hf\|_{(0,12t^*]}, & a>0,\end{cases}$$

where

$$t^*=\begin{cases}(rt)^{\frac{1}{1-\lambda}}, & 0<t<\frac{1}{8r},\ 0\le\lambda<1,\\ 0, & \lambda=1,\end{cases}\qquad \varphi(x)=x,\ w(x)=x^a(1+x)^b,\quad (a\ge0,b\in R).$$

$$\Omega^r_{\varphi^\lambda}(f,t)_w=\sup_{o<h\le t}\|w\Delta^r_{h\varphi^\lambda}f\|_{[t^*,\infty)}\qquad 0\le\lambda<1.$$

$$K^r_{\varphi^\lambda}(f,t^r)_w=\inf_g\big\{\|w(f-g)\|+t^r\|w\varphi^{r\lambda}g^{(r)}\|,\ g^{(r-1)}\in A.C._{loc}\big\}.$$

It is know that (cf. [2])

$$\omega^r_{\varphi^\lambda}(f,t)_w\sim K^r_{\varphi^\lambda}(f,t^r)_w, \tag{2.17}$$

$$C^{-1}\Omega^r_{\varphi^\lambda}(f,t)_w\le\omega^r_{\varphi^\lambda}(f,t)_w\le C\int_0^t\frac{\Omega^r_{\varphi^\lambda}(f,\tau)_w}{\tau}d\tau. \tag{2.18}$$

3 The Direct Theorem

In this section we will show the approximation direct theorem for $G_n^{(2r-1)}f$ with $2r$th Ditzian-Totik weighted modulus of smoothness.

Theorem 3.1. *Let $n \geq 4r$. Then for $wf \in L_\infty[0,\infty)$ we have*

$$\big|w(x)\big(G_n^{(2r-1)}(f,x)-f(x)\big)\big| \leq C\omega_{\varphi^\lambda}^{2r}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big)_w. \tag{3.1}$$

Proof. For any $g \in w_\infty^{2r} =: \big\{g : g^{(2r-1)} \in A.C._{loc},\ \ w\varphi^{2r\lambda}g^{(2r)} \in L_\infty\big\}$, by Taylor's formula

$$g(t) = \sum_{j=0}^{2r-1}\frac{1}{j!}(t-x)^j g^{(j)}(x) + R_{2r}(g,t,x) \tag{3.2}$$

with the integral remainder

$$R_{2r}(g,t,x) = \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1}g^{(2r)}(u)du. \tag{3.3}$$

As $G_n^{(2r-1)}(f,x)$ is exact on Π_{2r-1} and $\alpha_0^n = 1$, $\alpha_1^n = 0$ and (2.3) we have

$$\begin{aligned}
&\big|w(x)\big(G_n^{(2r-1)}(g,x)-g(x)\big)\big| \\
\leq&\big|wG_n\big(R_{2r}(g,\cdot,x),x\big)\big| + C\sum_{j=2}^{2r-1} n^{-\frac{j}{2}}\varphi^j(x)w(x)\big|D^jG_n\big(R_{2r}(g,\cdot,x),x\big)\big| \qquad (3.4)\\
=:&I_1+I_2.
\end{aligned}$$

For u between x and t, $\varphi(x) = x$ we have (cf. [2, Lemma 9.6.1])

$$\frac{|u-x|}{\varphi^\lambda(u)} \leq \frac{|x-t|}{\varphi^\lambda(x)}, \quad \frac{1}{\varphi^\lambda(u)} \leq \frac{1}{\varphi^\lambda(x)} + \frac{1}{\varphi^\lambda(t)}.$$

Therefore

$$|R_{2r}(g,t,x)| \leq C\frac{|t-x|^{2r-1}}{\varphi^{(2r-1)\lambda}(x)}\Big(\frac{1}{x^\lambda}+\frac{1}{t^\lambda}\Big)\|w\varphi^{2r\lambda}g^{(2r)}\|\Big|\int_x^t w^{-1}(u)du\Big|. \tag{3.5}$$

By (1.1), (2.5) and (3.5), one has

$$\begin{aligned}
&\big|w(x)D^jG_n(R_{2r}(g,\cdot,x),x)\big| \\
=&\big|w(x)\int_0^\infty \frac{\partial^j}{\partial x^j}g_n(x,t)R_{2r}\big(g,\frac{n}{t},x\big)dt\big| \\
=&\big|w(x)\frac{j!}{x^j}\int_0^\infty \frac{x^{n+1}}{n!}e^{-xt}t^nL_j^{(n+1-j)}(xt)R_{2r}\big(g,\frac{n}{t},x\big)dt\big| \\
=&\big|w(x)\frac{j!}{x^j}\int_0^\infty \frac{1}{n!}e^{-u}u^nL_j^{(n+1-j)}(u)R_{2r}\big(g,\frac{nx}{u},x\big)du\big| \\
\leq&C\big\|w\varphi^{2r\lambda}g^{(2r)}\big\|\frac{1}{x^j}\int_0^\infty \frac{1}{n!}e^{-u}u^n\Big|L_j^{(n+1-j)}(u)\Big|\frac{\big|\frac{nx}{u}-x\big|^{2r-1}}{\varphi^{(2r-1)\lambda}(x)}\Big(\frac{1}{x^\lambda}+\big(\frac{u}{nx}\big)^\lambda\Big)\Big|\int_x^{\frac{nx}{u}}\frac{w(x)}{w(\tau)}d\tau\Big|du.
\end{aligned}$$

Utilizing (2.9) we have

$$\begin{aligned}
\Big|\int_x^{\frac{nx}{u}}\frac{w(x)}{w(\tau)}d\tau\Big| &\leq C\Big|\int_x^{\frac{nx}{u}}\big(\frac{x}{\tau}\big)^a+\big(\frac{x}{\tau}\big)^{a+b}d\tau\Big| \\
&\leq C\Big(1+\big(\frac{u}{n}\big)^a+\big(\frac{u}{n}\big)^{a+b}\Big)\big|\frac{nx}{u}-x\big|.
\end{aligned}$$

Hence

$$\begin{aligned}&\big|w(x)D^jG_n(R_{2r}(g,\cdot,x),x)\big|\\ \le&C\|w\varphi^{2r\lambda}g^{(2r)}\|\frac{1}{x^{j+2r\lambda}}\frac{1}{n!}\int_0^\infty e^{-u}u^n\Big|L_j^{(n+1-j)}(u)\Big|\big(\frac{nx}{u}-x\big)^{2r}\\ &\times\Big(1+\big(\frac{u}{n}\big)^\lambda+\big(\frac{u}{n}\big)^a+\big(\frac{u}{n}\big)^{\lambda+a}+\big(\frac{u}{n}\big)^{a+b}+\big(\frac{u}{n}\big)^{\lambda+a+b}\Big)du.\end{aligned}\tag{3.6}$$

Utilizing (2.7) and (2.8) for $\forall\ \beta\in R$

$$\begin{aligned}&\int_0^\infty e^{-u}u^n\Big|L_j^{(n+1-j)}(u)\Big|\big(\frac{nx}{u}-x\big)^{2r}\big(\frac{u}{n}\big)^\beta du\\ \le&\Big(\int_0^\infty e^{-u}u^{n+1-j}\Big|L_j^{(n+1-j)}(u)\Big|^2du\Big)^{\frac12}\Big(\int_0^\infty e^{-u}u^{n+j-1}\big(\frac{nx}{u}-x\big)^{4r}\big(\frac{u}{n}\big)^{2\beta}du\Big)^{\frac12}\\ \le&\Big(\frac{\Gamma(n+2)}{j!}\Big)^{\frac12}\Big(\int_0^\infty e^{-u}u^{n+j-1}\big(\frac{nx}{u}-x\big)^{8r}du\Big)^{\frac14}\Big(C\int_0^\infty e^{-u}u^{n+j-1}\big(\frac{u}{u+j-1}\big)^{4\beta}du\Big)^{\frac14}\\ \le&C\big((n+1)!\big)^{\frac12}\big((n+j-1)!\frac{x^{8r}}{n^{4r}}\big)^{\frac14}\big((n+j-1)!\big)^{\frac14}.\end{aligned}\tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\begin{aligned}I_2\le&C\sum_{j=2}^{2r-1}n^{-\frac{j}{2}}x^j\frac{1}{x^{j+2r\lambda}}\frac{1}{n!}\big((n+1)!\big)^{\frac12}\big((n+j-1)!\big)^{\frac12}\frac{x^{2r}}{n^r}\big\|w\varphi^{2r\lambda}g^{(2r)}\big\|\\ \le&C\frac{x^{2r(1-\lambda)}}{n^r}\big\|w\varphi^{2r\lambda}g^{(2r)}\big\|.\end{aligned}\tag{3.8}$$

From the procedure of the proof of (3.8), similarly we can deduce that

$$I_1\le C\frac{x^{2r(1-\lambda)}}{n^r}\big\|w\varphi^{2r\lambda}g^{(2r)}\big\|.\tag{3.9}$$

Combining (3.4), (3.8) and (3.9) we have for $g\in w_\infty^{2r}$.

$$\big|w(x)\big(G_n^{(2r-1)}(g,x)-g(x)\big)\big|\le C\frac{\varphi^{2r(1-\lambda)}(x)}{n^r}\big\|w\varphi^{2r\lambda}g^{(2r)}\big\|.\tag{3.10}$$

By the definition of K-functional and (2.17) for $wf\in L_\infty$ we can choose $g=g_{n,x,\lambda}\in w_\infty^{2r}$ such that

$$\|w(f-g)\|+\frac{\varphi^{2r(1-\lambda)}(x)}{n^r}\|w\varphi^{2r\lambda}g^{(2r)}\|\le C\omega_{\varphi^\lambda}^{2r}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big)_w.\tag{3.11}$$

Then we have by (3.10)

$$\begin{aligned}&\big|w(x)\big(G_n^{(2r-1)}(f,x)-f(x)\big)\big|\\ \le&C\big(\|w(f-g)\|+\big|w\big(G_n^{(2r-1)}(g,x)-g(x)\big)\big|\big)\\ \le&C\big(\|w(f-g)\|+\frac{\varphi^{2r(1-\lambda)}(x)}{n^r}\|w\varphi^{2r\lambda}g^{(2r)}\|\big)\\ \le&C\omega_{\varphi^\lambda}^{2r}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big)_w.\end{aligned}$$

Now (3.1) is proved. □

Remark. In the proof of (3.1) we use (2.1), (2.2), (2.3) and (2.7) on the Laguerre polynomial. On the other way we also can use the formula (cf. [2, (9.4.11)]

$$\big(G_n(f,x)\big)^{(r)}=\sum_{i=0}^r Q_i(n,x)G_n\big((t-x)^if(t),x\big),$$

where $Q_i(n,x)=\sum\limits_{2j+l-i=r} C(i,l)\frac{n^j}{x^{2j+l}}$ and so $x^r\|Q_i(n,x)|\le C\frac{n^{\frac{r+i}{n}}}{x^i}$. Thus it need not use the Laguerre polynomial in the proof of (3.1). In the next section the case is similar.

4 The Inverse Theorem

To prove the inverse theorem we need a new K- functional, for this reason we introduce some notations. For $0\le\lambda\le 1$, $0<\alpha<2r$, we define

$$\begin{aligned}
\|f\|_0 &= \sup_{x\in(0,\infty)}\big|w(x)\varphi^{\alpha(\lambda-1)}(x)f(x)\big|,\\
C^0_{\lambda,w} &= \big\{f\big|wf\in L_\infty,\ \|f\|_0<\infty\big\},\\
\|f\|_{2r} &= \sup_{x\in(0,\infty)}\big|w(x)\varphi^{2r+\alpha(\lambda-1)}(x)f^{(2r)}(x)\big|,\\
C^{2r}_{\lambda,\omega} &= \big\{f\in C^0_{\lambda,\omega}: f^{(2r-1)}\in A.C._{loc},\ \|f\|_{2r}<\infty\big\}.
\end{aligned}$$

Now we give a new K-functional

$$K^\alpha_\lambda(f,t^{2r})_w=\inf_{g\in C^{2r}_{\lambda,\omega}}\{\|f-g\|_0+t^{2r}\|g\|_{2r}\}. \tag{4.1}$$

The next lemma shows two inequalities which will be used.

Lemma 4.1. *For $n\ge 4r$ we have*

$$\|G^{(2r-1)}_n f\|_{2r}\le Cn^r\|f\|_0,\qquad f\in C^0_{\lambda,\omega}, \tag{4.2}$$

$$\|G^{(2r-1)}_n f\|_{2r}\le C\|f\|_{2r},\qquad f\in C^{2r}_{\lambda,\omega}. \tag{4.3}$$

Proof. At first we prove (4.1). Duo to [8, (32)] we have

$$\begin{aligned}
&\big|w(x)\varphi^{2r+\alpha(\lambda-1)}(x)\big(G^{(2r-1)}_n(f,x)\big)^{(2r)}\big|\\
=&\Big|w(x)\varphi^{2r+\alpha(\lambda-1)}(x)\Big((G_nf)^{(2r)}(x)+\Big(\sum_{j=2}^{2r-1}\frac{1}{n^j}L_j^{(n-j)}(n)\varphi^j(x)D^jG_n(f,x)\Big)^{(2r)}\Big)\Big|\\
\le&\Big|w(x)\varphi^{2r+\alpha(\lambda-1)}(x)(G_nf)^{(2r)}(x)\Big|\\
&\quad+Cw(x)\varphi^{2r+\alpha(\lambda-1)}(x)\sum_{j=2}^{2r-1}n^{-\frac{j}{2}}\sum_{k=0}^{j}\big|\varphi^{j-k}(x)(G_nf)^{(2r+j-k)}(x)\big|\\
=:&J_1+J_2.
\end{aligned} \tag{4.4}$$

By (1.1) and (2.5) we have

$$\begin{aligned}
(G_nf)^{(2r+j-k)}(x)&=\int_0^\infty\frac{\partial^{2r+j-k}}{\partial x^{2r+j-k}}g_n(x,t)f\big(\frac{n}{t}\big)dt\\
&=\frac{(2r+j-k)!}{x^{2r+j-k}}\frac{1}{n!}\int_0^\infty x^{n+1}e^{-tx}t^nL^{(n+1-2r-j+k)}_{2r+j-k}(xt)f\big(\frac{n}{t}\big)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
J_2 &\le C\sum_{j=2}^{2r-1} n^{-\frac{j}{2}} w(x)\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{j}\frac{1}{n!}\int_0^\infty e^{-u}u^n\big|L_{2r+j-k}^{(n+1-2r-j+k)}(u)\big|w^{-1}\big(\frac{nx}{u}\big)\varphi^{\alpha(1-\lambda)}\big(\frac{nx}{u}\big)du\|f\|_0\\
&\le C\sum_{j=2}^{2r-1} n^{-\frac{j}{2}}\|f\|_0\sum_{k=0}^{j}\frac{1}{n!}\big((n+1)!\big)^{\frac12}\Big(\int_0^\infty e^{-u}u^{n-1+2r+j-k}\frac{w^2(x)\varphi^{2\alpha(\lambda-1)}(x)}{w^2\big(\frac{nx}{u}\big)\varphi^{2\alpha(\lambda-1)}\big(\frac{nx}{u}\big)}du\Big)^{\frac12}\\
&\le C\sum_{j=2}^{2r-1} n^{-\frac{j}{2}}\|f\|_0\sum_{k=0}^{j}\frac{1}{n!}\big((n+1)!\big)^{\frac12}\big((n-1+2r+j-k)!\big)^{\frac12}\\
&\le Cn^r\|f\|_0.
\end{aligned}$$

Similarly we also have

$$J_1 \le Cn^r\|f\|_0.$$

Thus, (4.2) is valid. Now we prove (4.3).

In the same way, by (2.5) and (2.6) we have

$$\begin{aligned}
(G_nf)^{(2r+j-k)} &= \frac{n^{2r}(n-2r)!}{n!}\int_0^\infty\frac{(j-k)!}{x^{j-k}}g_{n-2r}(x,u)L_{j-k}^{(n-2r+1-j+k)}(xu)f^{(2r)}\big(\frac{n}{u}\big)du\\
&= \frac{n^{2r}(n-2r)!}{n!}\frac{(j-k)!}{x^{j-k}}\frac{1}{(n-2r)!}\int_0^\infty e^{-t}t^{n-2r}L_{j-k}^{(n-2r+1-j+k)}(t)f^{(2r)}\big(\frac{nx}{t}\big)dt.
\end{aligned}$$

Similarly from above procedure we can deduce

$$J_2 \le C\|f\|_{2r},$$

$$J_1 \le C\|f\|_{2r},$$

and so (4.3) is proved. □

Theorem 4.2. *For $wf \in L_\infty$, $0 \le \lambda \le 1$, $0 < \alpha < 2r$, $n \ge 4r$, then*

$$\big|w\big(G_n^{(2r-1)}(f,x) - f(x)\big)\big| \le O\Big(\big(\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\big)^\alpha\Big) \tag{4.5}$$

implies

$$\omega^{2r}_{\varphi^\lambda}(f,t)_w = O\big(t^\alpha\big). \tag{4.6}$$

Proof. By the definition of $K^\alpha_\lambda(f,t^{2r})_w$, for a fixed x and λ we can choose $g \in C^{2r}_{\lambda,w}$, such that

$$\|f-g\|_0 + n^{-r}\|g\|_{2r} \le 2K^\alpha_\lambda(f,n^{-r})_w. \tag{4.7}$$

By (4.5) we have

$$\big|w(x)\varphi^{\alpha(\lambda-1)}(x)\big(G_n^{(2r-1)}(f,x) - f(x)\big| \le Cn^{-\frac{\alpha}{2}}. \tag{4.8}$$

Utilizing Lemma 4.1 we obtain

$$\begin{aligned}
K^\alpha_\lambda(f,t^{2r})_w &\le \|f - G_n^{(2r-1)}f\|_0 + t^{2r}\|G_n^{(2r-1)}f\|_{2r}\\
&\le Cn^{-\frac{\alpha}{2}} + t^{2r}\big(\|G_n^{(2r-1)}(f-g)\|_{2r} + \|G_n^{(2r-1)}g\|_{2r}\big)\\
&\le C\big(n^{-\frac{\alpha}{2}} + t^{2r}(n^r\|f-g\|_0 + \|g\|_{2r})\big)\\
&\le C\big(n^{-\frac{\alpha}{2}} + t^{2r}n^rK^\alpha_\lambda(f,n^{-r})_w\big).
\end{aligned}$$

By Berence-Lorentz Lemma we get

$$K_\lambda^\alpha(f,t^{2r})_w = O(t^\alpha). \tag{4.9}$$

When $\lambda = 1$, $K_1^\alpha(f,t^{2r})_w = K_\varphi^{2r}(f,t^{2r})_w = O(t^\alpha)$.

So that

$$\omega_\varphi^{2r}(f,t)_w = O(t^\alpha). \tag{4.10}$$

When $0 \le \lambda < 1$, $x \ge t^* = (2rt)^{\frac{1}{1-\lambda}}$, $x - rh\varphi^\lambda(x) \ge 0$, then (cf. [2, p21, p27])

$$\frac{x}{2} \le x + (j-r)h\varphi^\lambda(x) \le 2x, \quad j = 0,1,\cdots,2r, \quad h \le \frac{1}{r}.$$

Therefore for $u \in [-rh\varphi^\lambda(x), rh\varphi^\lambda(x)]$ we have

$$\varphi(x+u) \sim \varphi(x), \qquad w(x+u) \sim w(x).$$

Then

$$w(x)\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \cdots \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-2r+\alpha(1-\lambda)}\big(x+\sum_{i=1}^{2r}u_i\big)w^{-1}\big(x+\sum_{i=1}^{2r}u_i\big)du_1\cdots du_{2r} \le Ch^{2r}\frac{\varphi^{\alpha(1-\lambda)}(x)}{\varphi^{2r(1-\lambda)}(x)}.$$

For g in (4.7) and $x \ge t^*$ we have

$$\begin{aligned}
&w(x)\big|\Delta_{h\varphi^\lambda(x)}^{2r}f(x)\big| \\
\le& w(x)\big|\Delta_{h\varphi^\lambda(x)}^{2r}(f-g)(x)\big| + w(x)\big|\Delta_{h\varphi^\lambda}^{2r}g(x)\big| \\
=& w(x)\big|\sum_{k=0}^{2r}(-1)^k\binom{2r}{k}(f-g)(x+(r-k)h\varphi^\lambda(x))\big| \\
&+ w(x)\Big|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\cdots\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} g^{(2r)}\big(x+\sum_{i=1}^{2r}u_i\big)du_1\cdots du_{2r}\Big| \\
\le& C\varphi^{\alpha(1-\lambda)}(x)\big(\|f-g\|_0 + \frac{h^{2r}}{\varphi^{2r(1-\lambda)}(x)}\|g\|_{2r}\big) \\
\le& C\varphi^{\alpha(1-\lambda)}(x)K_\lambda^\alpha\Big(f,\frac{h^{2r}}{\varphi^{2r(1-\lambda)}(x)}\Big)_w \\
\le& Ch^\alpha.
\end{aligned}$$

Thus by the definition of $\Omega_{\varphi^\lambda}^{2r}(f,t)_w$, we have

$$\Omega_{\varphi^\lambda}^{2r}(f,t)_w = O(t^\alpha). \tag{4.11}$$

From (2.18) and (4.11) we have

$$\omega_{\varphi^\lambda}^{2r}(f,t)_w = O(t^\alpha).$$

The proof of Theorem 4.2 is complete. □

Remark. By Theorem 3.1 and 4.2, the weighted approximation equivalent theorem (1.5) for $G_n^{(2r-1)}(f,x)$ is valid.

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Natural Splines of Birkhoff Type and Optimal Approximation

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1 Preliminaries

Let us consider an arbitrary finite interval $[a,b]$, $a<b$, on the real line and the Lebesgue space $L^2[a,b]$ with the usual inner product

$$< f,g >_2 := \int_a^b f(x)g(x)dx \tag{1}$$

and the corresponding norm

$$\|f\|_2^2 := \int_a^b f^2(x)dx. \tag{2}$$

We denote by $H^{m,2}[a,b]$ the linear space of all functions $f:[a,b]\to I\!R$ which satisfy the following conditions:

i) $f\in C^{m-1}[a,b]$,

ii) $f^{(m-1)}$ is absolutely continuous,

iii) $f^{(m)}\in L^2[a,b]$,

endowed with the norm

$$\|f\|_{m,2}^2 := \left\|f^{(m)}\right\|_2^2 + \sum_{k=0}^{m-1}\left[f^{(k)}(a)\right]^2. \tag{3}$$

Let us take $x_1,x_2,...,x_r$ as distinct knots in the interval $[a,b]$, $a\le x_1<x_2<...<x_r\le b$, the numbers $\alpha_1,\alpha_2,...,\alpha_r\in I\!N$, where $1\le\alpha_i\le m$, $i=1,\dots,r$, and the sets $I_i\subseteq\{0,1,...,\alpha_i-1\}$, $i=1,\dots,r$.

Of importance is the number of interpolation conditions, namely

$$n := \sum_{i=1}^{r} card(I_i). \tag{4}$$

Definition 1 *The set*

$$\Lambda := \left\{\lambda_{i,\nu_i} : H^{m,2}[a,b]\to I\!R,\ \lambda_{i,\nu_i}(f)=f^{(\nu_i)}(x_i),\ i=1,\dots,r,\ \nu_i\in I_i\right\} \tag{5}$$

is named a set of Birkhoff-type functionals on $H^{m,2}[a,b]$.

Definition 2 *For each* $y=(y_1,...,y_n)\in \mathbb{R}^n$ *we define the Birkhoff-type interpolatory set*

$$U_y := \left\{u\in H^{m,2}[a,b] | u^{(\nu_i)}(x_i)=y_{i,\nu_i},\ i=1,\dots,r,\ \nu_i\in I_i\right\}, \tag{6}$$

where

$$y := (y_1,...,y_n) = ((y_{1,\nu_1})_{\nu_1\in I_1},...,(y_{r,\nu_r})_{\nu_r\in I_r}). \tag{7}$$

Definition 3 *The problem of finding functions* $s\in U_y$ *which satisfy*

$$\left\|s^{(m)}\right\|_2 = \min_{u\in U_y}\left\|u^{(m)}\right\|_2 \tag{8}$$

is called Birkhoff-type natural spline interpolation problem, corresponding to the interpolatory set U_y.

One such s, *provided it exists, is called Birkhoff-type natural spline interpolation function, corresponding to the interpolatory set* U_y.

We denote by S_Λ the set of all solutions of the problems (8), where $y\in\mathbb{R}^n$.

Lemma 1 $s\in S_\Lambda$ *if and only if* s *satisfies the following conditions*:

i) $s^{(2m)}=0$, *for* $x\in(x_1,x_2)\cup...\cup(x_{r-1},\ x_r)$,

ii) $s^{(m)}=0$, *for* $x\in[a,x_1)\cup(x_r,b]$,

iii) a) $s^{(j)}(x_i-0)=s^{(j)}(x_i+0),\ i=1,\dots,r,\ j=0,\dots,m-1$,

b) $s^{(2m-1-\mu_i)}(x_i-0)=s^{(2m-1-\mu_i)}(x_i+0),\ i=1,\dots,r,\ \mu_i\in\{0,\dots,m-1\}\setminus I_i$.

For a proof see [3].

In the sequel we assume that $n\geq m$ and $\Pi_{m-1}\cap U_0=\{0\}$, where U_0 is the interpolatory set corresponding to $y_0=(0,...,0)\in\mathbb{R}^n$. If Λ contains at least m functionals of Hermite-type, then $\Pi_{m-1}\cap U_0=\{0\}$ (for a proof see [3]).

Lemma 2 *For each* $y\in\mathbb{R}^n$, *if the set* U_y *is nonempty, then problem* (8) *(corresponding to* U_y*) has unique solution* s_y *given by*

$$s_y(x)=\sum_{k=0}^{m-1}a_k^{(y)}\frac{(b-x)^k}{k!}+\sum_{i=1}^{r}\sum_{\nu_i\in I_i}b_{i,\nu_i}^{(y)}\frac{(x-x_i)_+^{2m-1-\nu_i}}{(2m-1-\nu_i)!}, \tag{9}$$

where the coefficients $a_k^{(y)}$, $k=0,\dots,m-1$, $b_{i,\nu_i}^{(y)}$, $i=1,\dots,r$, $\nu_i\in I_i$, *are given by the following linear system*

$$\begin{cases} \displaystyle\sum_{k=\mu_1}^{m-1}a_k^{(y)}(-1)^{\mu_1}\frac{(b-x_1)^{k-\mu_1}}{(k-\mu_1)!}=y_{1,\mu_1}, \quad \mu_1\in I_1, \\ \displaystyle\sum_{k=\mu_j}^{m-1}a_k^{(y)}(-1)^{\mu_j}\frac{(b-x_j)^{k-\mu_j}}{(k-\mu_j)!}+\sum_{i=1}^{j-1}\sum_{\nu_i\in I_i}b_{i,\nu_i}^{(y)}\frac{(x_j-x_i)^{2m-1-\nu_i-\mu_j}}{(2m-1-\nu_i-\mu_j)!}=y_{j,\mu_j}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=2,\dots,r, \quad \mu_j\in I_j, \\ \displaystyle\sum_{i=1}^{r}\sum_{\substack{\nu_i\in I_i\\ \nu_i\leq l}}b_{i,\nu_i}^{(y)}\frac{(b-x_i)^{l-\nu_i}}{(l-\nu_i)!}=0, \quad l=0,\dots,m-1. \end{cases} \tag{10}$$

The proof follows directly from Lemma 1.

Remark 1 *Lemma 2 implies that the matrix P_B of the system* (10) *is nonsingular.*

Definition 4 *For each $j = 1, \ldots, r$ and $\mu_j \in I_j$, let $y_{j,\mu_j} \in \mathbb{R}^n$ be defined by*

$$y_{j,\mu_j} := (\delta_{ij}\delta_{\nu_i,\mu_j})_{i=1,\ldots,r,\ \nu_i \in I_i} = \left((\delta_{1j}\delta_{\nu_1,\mu_j})_{\nu_1 \in I_1}, \ldots, (\delta_{rj}\delta_{\nu_r,\mu_j})_{\nu_r \in I_r}\right) \tag{11}$$

and the corresponding interpolatory set

$$U_{j,\mu_j} := \left\{u \in H^{m,2}[a,b] | u^{(\nu_i)}(x_i) = \delta_{ij}\delta_{\nu_i,\mu_j},\ i = 1, \ldots, r,\ \nu_i \in I_i\right\}. \tag{12}$$

Lemma 3 *For each $j = 1, \ldots, r$ and $\mu_j \in I_j$, if the set U_{j,μ_j} is nonempty, then the corresponding problem* (8) *has a unique solution*

$$s_{j,\mu_j}(x) = \sum_{k=0}^{m-1} a_k^{(j,\mu_j)} \frac{(b-x)^k}{k!} + \sum_{i=1}^{r} \sum_{\nu_i \in I_i} b_{i,\nu_i}^{(j,\mu_j)} \frac{(x-x_i)_+^{2m-1-\nu_i}}{(2m-1-\nu_i)!}, \tag{13}$$

where the coefficients $a_k^{(j,\mu_j)}$, $k = 0, \ldots, m-1$, $b_{i,\nu_i}^{(j,\mu_j)}$, $i = 1, \ldots, r$, $\nu_i \in I_i$, are given by the system of equations

$$P_B \left(a_0^{(j,\mu_j)}, \ldots, a_{m-1}^{(j,\mu_j)}, \left(b_{1,\nu_1}^{(j,\mu_j)}\right)_{\nu_1 \in I_1}, \ldots, \left(b_{r,\nu_r}^{(j,\mu_j)}\right)_{\nu_r \in I_r}\right)^t \tag{14}$$

$$= \left((\delta_{1j}\delta_{\nu_1,\mu_j})_{\nu_1 \in I_1}, \ldots, (\delta_{rj}\delta_{\nu_r,\mu_j})_{\nu_r \in I_r}, \underbrace{0, \ldots, 0}_{m}\right)^t.$$

This result is a consequence of Lemma 2.

In the sequel we assume that the sets U_{j,μ_j}, $j = 1, \ldots, r$, $\mu_j \in I_j$, are nonempty.

Definition 5 *s_{j,μ_j}, $j = 1, \ldots, r$, $\mu_j \in I_j$, defined by (13), are the fundamental Birkhoff-type natural spline interpolation functions.*

Remark 2 *The functions s_{j,μ_j}, $j = 1, \ldots, r$, $\mu_j \in I_j$, are characterized in S_Λ by the properties*

$$s_{j,\mu_j}^{(\nu_i)}(x_i) = \delta_{ij}\delta_{\nu_i,\mu_j},\ i = 1, \ldots, r,\ \nu_i \in I_i. \tag{15}$$

Lemma 4 *For each $y \in \mathbb{R}^n$ the set U_y is nonempty and the problem* (8)*, corresponding to U_y, has unique solution given by*

$$s_y = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} y_{i,\nu_i} s_{i,\nu_i}, \tag{16}$$

where

$$y := (y_1, \ldots, y_n) = ((y_{1,\nu_1})_{\nu_1 \in I_1}, \ldots, (y_{r,\nu_r})_{\nu_r \in I_r}). \tag{17}$$

The proof is a consequence of Lemma 2, Lemma 3 and relation (15).

Definition 6 *With each $f \in H^{m,2}[a,b]$ we associate $y_f \in \mathbb{R}^n$,*

$$y_f := \left(f^{(\nu_i)}(x_i)\right)_{i=1,\ldots,r,\ \nu_i \in I_i} = \left(\left(f^{(\nu_1)}(x_1)\right)_{\nu_1 \in I_1}, \ldots, \left(f^{(\nu_r)}(x_r)\right)_{\nu_r \in I_r}\right), \tag{18}$$

and the corresponding interpolatory set

$$U_{y_f} := \left\{ u \in H^{m,2}[a,b] | u^{(\nu_i)}(x_i) = f^{(\nu_i)}(x_i),\ i = 1, \ldots, r,\ \nu_i \in I_i \right\}. \tag{19}$$

Remark 3 *The interpolatory set U_{y_f} is nonempty ($f \in U_{y_f}$).*

Remark 4 *Lemma 2 implies that the problem (8), corresponding to U_{y_f}, has unique solution for each $f \in H^{m,2}[a,b]$.*

Definition 7 *The operator $S : H^{m,2}[a,b] \to S_\Lambda$, where Sf is the unique solution of the problem (8), corresponding to the set U_{y_f}, i.e.,*

$$\left\| (Sf)^{(m)} \right\|_2 = \inf_{u \in U_{y_f}} \left\| u^{(m)} \right\|_2, \tag{20}$$

is called Birkhoff-type natural spline interpolation operator.

The formula

$$f = Sf + Rf, \quad f \in H^{m,2}[a,b], \tag{21}$$

is termed Birkhoff-type natural spline interpolation formula, where the operator

$$R : H^{m,2}[a,b] \to U_0$$

is the remainder operator.

Remark 5 *Lemma 4 implies that formula (21) can be written as*

$$f = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} f^{(\nu_i)}(x_i) s_{i,\nu_i} + Rf, \quad f \in H^{m,2}[a,b]. \tag{22}$$

Remark 6 *From Lemma 4 we obtain that the Birkhoff-type natural spline interpolation formula is exact for every $s \in S_\Lambda$, i.e.,*

$$Rs = 0, \quad \text{for all } s \in S_\Lambda. \tag{23}$$

Lemma 5 *For the remainder operator R we have the expression*

$$(Rf)(x) = \int_a^b \varphi(x,t) f^{(m)}(t) dt, \quad f \in H^{m,2}[a,b], \tag{24}$$

where

$$\varphi(x,t) := R^x \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right] = \frac{(x-t)_+^{m-1}}{(m-1)!} - \sum_{i=1}^{r} \sum_{\nu_i \in I_i} \frac{(x_i - t)_+^{m-1-\nu_i}}{(m-1-\nu_i)!} s_{i,\nu_i} \tag{25}$$

(R^x means that R is acting on the variable x).

The proof is obtained from the Peano Theorem, noticing that $Rs = 0$, $s \in S_\Lambda$, and $\Pi_{m-1} \subset S_\Lambda$.

2 Main result

Let us take a linear functional $\lambda : H^{m,2}[a,b] \to I\!R$ which satisfies the condition

$$\lambda^x \left(\int_a^b \frac{(x-t)_+^{m-1}}{(m-1)!} v(t)dt \right) = \int_a^b \lambda^x \left(\frac{(x-t)_+^{m-1}}{(m-1)!} \right) v(t)dt, \quad \text{for all } v \in L^2[a,b]. \tag{26}$$

(Again λ^x means that λ is acting on the variable x).

We suppose that the functionals $\lambda_{i,\nu_i} \in \Lambda$, $i = 1, \ldots, r$, $\nu_i \in I_i$, being of Birkhoff-type, and the functional λ are linearly independent.

Definition 8 *An optimal approximation formula of Sard-type, corresponding to the functional λ and the set $\Lambda = \{\lambda_{i,\nu_i},\ i = 1, \ldots, r,\ \nu_i \in I_i\}$ of Birkhoff-type functionals, is a formula of the form*

$$\lambda(f) = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} A^*_{i,\nu_i} f^{(\nu_i)}(x_i) + R^*(f), \tag{27}$$

which satisfies the conditions

$$\begin{aligned} &i) \quad R^*(e_i) = 0, \quad i = 0, \ldots, m-1, \\ &ii) \quad \int_a^b [K^*(t)]^2 \, dt \to min, \end{aligned} \tag{28}$$

where

$$\begin{aligned} K^*(t) &:= R^{*x} \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right] \\ &= \lambda^x \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right] - \sum_{i=1}^{r} \sum_{\nu_i \in I_i} A^*_{i,\nu_i} \frac{(x_i-t)_+^{m-1-\nu_i}}{(m-1-\nu_i)!}. \end{aligned} \tag{29}$$

Lemma 6 *The remainder $R^* : H^{m,2}[a,b] \to I\!R$ has the following properties*

i) $R^*(P) = 0, \quad \text{for all} \quad P \in \Pi_{m-1}$,

ii) $R^*(f) = \int_a^b K^*(t) f^{(m)}(t)dt, \quad f \in H^{m,2}[a,b]$.

The proof follows directly from Definition 8, taking into account that λ satisfies condition (26).

Theorem 1 *Let us consider the Birkhoff-type natural spline interpolation formula* (22)*, corresponding to the set Λ, i.e.,*

$$f = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} f^{(\nu_i)}(x_i) s_{i,\nu_i} + Rf, \quad f \in H^{m,2}[a,b].$$

Applying λ on both sides of this equality we obtain

$$\lambda(f) = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} \bar{A}_{i,\nu_i} f^{(\nu_i)}(x_i) + \bar{R}(f), \quad f \in H^{m,2}[a,b], \tag{30}$$

where

$$\bar{A}_{i,\nu_i} = \lambda(s_{i,\nu_i}), \quad i = 1, \ldots, r, \quad \nu_i \in I_i, \tag{31}$$

$$\bar{R}(f) = \lambda(Rf), \quad f \in H^{m,2}[a,b]. \tag{32}$$

Then

$$\bar{R}(f) = \int_a^b \bar{K}(t) f^{(m)}(t)dt, \tag{33}$$

where

$$\bar{K}(t) := \lambda^x[\varphi(x,t)] = \bar{R}^x\left[\frac{(x-t)_+^{m-1}}{(m-1)!}\right], \tag{34}$$

and (30) *is the unique optimal approximation formula of Sard-type, corresponding to the functional* λ *and the set* Λ *of Birkhoff-type functionals.*

Proof. Lemma 5 and relation (32) imply that

$$\bar{R}(f) = \lambda(Rf) = \lambda^x\left(\int_a^b \varphi(x,t) f^{(m)}(t)dt\right). \tag{35}$$

Taking into account that λ satisfies condition (26), after a rather simple computation, we obtain

$$\bar{R}(f) = \int_a^b \lambda^x[\varphi(x,t)] f^{(m)}(t)dt. \tag{36}$$

Denoting

$$\bar{K}(t) := \lambda^x[\varphi(x,t)] \tag{37}$$

and using relation (25) we deduce that the equalities (33) and (34) hold.

Let us show that (30) is the unique optimal approximation formula of Sard-type, i.e.,

$$\begin{aligned} &i) \quad \bar{R}(e_i) = 0, \quad i = 0, \dots, m-1, \\ &ii) \quad \int_a^b \left[\bar{K}(t)\right]^2 dt \to min. \end{aligned} \tag{38}$$

Relation (33) implies that

$$\bar{R}(e_i) = \int_a^b \bar{K}(t) e_i^{(m)}(t)dt = 0, \quad i = 0, \dots, m-1. \tag{39}$$

Therefore (38) i) holds.

For (38) ii) let us suppose that the optimal approximation formula of Sard-type would be

$$\lambda(f) = \sum_{i=1}^{r} \sum_{\nu_i \in I_i} A_{i,\nu_i}^* f^{(\nu_i)}(x_i) + R^*(f), \quad f \in H^{m,2}[a,b], \tag{40}$$

i.e.,

$$\begin{aligned} &i) \quad R^*(e_i) = 0, \quad i = 0, \dots, m-1, \\ &ii) \quad \int_a^b [K^*(t)]^2 dt \to min, \end{aligned} \tag{41}$$

where

$$\begin{aligned} K^*(t) &:= R^{*x}\left[\frac{(x-t)_+^{m-1}}{(m-1)!}\right] \\ &= \lambda^x\left[\frac{(x-t)_+^{m-1}}{(m-1)!}\right] - \sum_{i=1}^{r} \sum_{\nu_i \in I_i} A_{i,\nu_i}^* \frac{(x_i-t)_+^{m-1-\nu_i}}{(m-1-\nu_i)!}. \end{aligned} \tag{42}$$

Consider the function

$$\sigma(t) = \bar{K}(t) - K^*(t) \tag{43}$$

and observe that by (34) and (42) it may be written as

$$\begin{aligned} \sigma(t) &= \bar{R}^x \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right] - R^{*x} \left[\frac{(x-t)_+^{m-1}}{(m-1)!} \right] \\ &= -\sum_{i=1}^{r} \sum_{\nu_i \in I_i} \left(\bar{A}_{i,\nu_i} - A^*_{i,\nu_i} \right) \frac{(x_i - t)_+^{m-1-\nu_i}}{(m-1-\nu_i)!}. \end{aligned} \tag{44}$$

It is obvious that the function $\sigma(t)$ satisfies the following conditions:

$$\begin{aligned} &i) \quad \sigma^{(m)}(t) = 0, \quad \text{for } t \in (x_1, x_2) \cup \ldots \cup (x_{r-1}, x_r), \\ &ii) \quad \sigma(t) = 0, \quad \text{for } t \in [a, x_1) \cup (x_r, b], \\ &iii) \quad \sigma^{(m-1-\mu_i)}(x_i - 0) = \sigma^{(m-1-\mu_i)}(x_i + 0), \ i = 1, \ldots, r, \ \mu_i \in \{0, \ldots, m-1\} \setminus I_i. \end{aligned} \tag{45}$$

Let us consider now a function $s \in H^{m,2}[a, b]$ which satisfies

$$s^{(m)}(t) = \sigma(t). \tag{46}$$

Using (45), (46) and taking into account that $s \in H^{m,2}[a, b]$ we obtain that s verifies the conditions $i), ii), iii)$ from Lemma 1, hence $s \in S_\Lambda$.

We know from Remark 6 that the remainder must vanish, i.e., $Rs = 0$, which implies that

$$\bar{R}(s) = 0. \tag{47}$$

Using (47) we conclude that

$$\int_a^b \bar{K}(t) \left[\bar{K}(t) - K^*(t) \right] dt = 0, \tag{48}$$

and a direct consequence of this is the relation

$$\int_a^b \left[K^*(t) \right]^2 dt = \int_a^b \left[\bar{K}(t) - K^*(t) \right]^2 dt + \int_a^b \left[\bar{K}(t) \right]^2 dt. \tag{49}$$

Relation (41) $ii)$ implies that

$$\int_a^b \left[K^*(t) \right]^2 dt \leq \int_a^b \left[\bar{K}(t) \right]^2 dt, \tag{50}$$

consequently from (49) we obtain the inequality

$$\int_a^b \left[\bar{K}(t) - K^*(t) \right]^2 dt \leq 0. \tag{51}$$

Using (51) we deduce that $\sigma(t)$ vanishes identically on $[a, b]$, therefore

$$\bar{A}_{i,\nu_i} = A^*_{i,\nu_i}, \quad i = 1, \ldots, r, \quad \nu_i \in I_i, \tag{52}$$

and

$$\bar{R}(f) = R^*(f). \tag{53}$$

This concludes the proof of Theorem 1.

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Asymptotic Expansion of a Sequence of Divided Differences with Application to Positive Linear Operators

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Abstract. We give a representation for divided differences of monomials in terms of exponential complete Bell polynomials. As an application we derive the complete asymptotic expansion for a sequence of positive linear operators approximating continuous functions on a finite interval.

Keywords: divided difference, asymptotic expansion, Bell polynomials

1. Introduction

2. A representation for divided differences of monomials

Let $z_0, \ldots, z_n$ be pairwise different points of the complex plane and $f : G \to \mathbb{C}$ be an arbitrary function whose domain G contains all z_j $(j = 0, \ldots, n)$. Denote by $[z_0, \ldots, z_n; f]$ the divided difference of the function f on the knots $z_0, \ldots, z_n$, given by

$$[z_0, \ldots, z_n; f] = \sum_{j=0}^{n} \frac{f(z_j)}{(z_j - z_0) \ldots (z_j - z_{j-1})(z_j - z_{j+1}) \ldots (z_j - z_n)}.$$

Consider the monomials $e_j : \mathbb{C} \to \mathbb{C}$, $e_j(z) = z^j$ $(j = 0, 1, \ldots)$. It is obvious that $[z_0, \ldots, z_n; e_j] = 0$ $(j = 0, \ldots, n-1)$ and $[z_0, \ldots, z_n; e_n] =$

1. In the case $j > n$, using the identity

$$\left[z_0, \ldots, z_n; \frac{1}{z - \cdot}\right] = \frac{1}{(z - z_0) \ldots (z - z_n)}, \tag{1}$$

Popoviciu [17] proved the formula

$$[z_0, \ldots, z_n; e_{n+r}] = \sum z_0^{k_0} \ldots z_n^{k_n}, \tag{2}$$

where the sum runs over all $k_0, \ldots, k_n \in \{0, 1, \ldots, r\}$ with $k_0 + \cdots + k_n = r$. Formula (2) was rediscovered in 1981 by Neuman [14].

Consider a triangular matrix of complex knots $(z_{n,k})$ $(k = 0, \ldots, n;$ $n = 0, 1, \ldots)$.

In 1995 Ivan and Raşa [11] obtained the first term of the asymptotic expansion for $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}]$ in the case when r is even and the knots $z_{n,k}$ are equidistant real numbers in $[-1, 1]$ satisfying the condition

$$z_{n,n-k} = -z_{n,k} \quad (n = 0, 1, \ldots;\ k = 0, 1, \ldots, n). \tag{3}$$

A more general system of knots is considered in [12]. In this paper we give a representation of $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}]$ in terms of exponential complete Bell polynomials without any restriction on the knots $z_{n,k}$.

Ivan and Raşa [11, 12] used the estimation obtained for $[z_{n,0}, \ldots,$ $z_{n,n};\ e_{n+r}]$ in order to study the asymptotic behaviour of the operators $L_n : C\,[-a - 1, a + 1] \to C\,[-a, a]\,,$

$$L_n(f; x) = n![x + h_0, \ldots, x + h_n; f^{(-n)}], \tag{4}$$

$a > 0$, $n = 1, 2, \ldots$, where $h_i = -1 + \frac{2i}{n}$, $i = 0, \ldots, n$, and $f^{(-n)}$ is an n-th antiderivative of f.

The operators (4) were considered by Zwick [25] and Pečarić and Raşa [20]. They also can be given by the n-fold integral

$$L_n\,(f; x) = 2^{-n} \int_{x-1}^{x+1} \ldots \int_{x-1}^{x+1} f\left(\frac{t_1 + \cdots + t_n}{n}\right) dt_1 \ldots dt_n \qquad [21].$$

The L_n are positive linear operators of probabilistic type and Bernstein-Schnabl type operators (cf. [21]). Various inequalities involving $L_n f$ have been studied in [18, 16, 25, 15, 19].

As positive operators, L_n have been studied in [20, 21]. For $f \in$ $C\,[-a - 1, a + 1]\,,$ the operators L_n verify:

$$\|L_n f - f\| \le 2\,\omega\left(f, \frac{1}{\sqrt{3n}}\right), \quad \|L_n f - f\| \le 2.25\,\omega_2\left(f, \frac{1}{\sqrt{3n}}\right), \text{ [20, 12]}$$

where ω and ω_2 denotes the usual and the second order modulus of continuity, respectively. Consequently, Eq. (4) defines a sequence of approximation operators.

If $f \in C[-a-1, a+1]$ admits a second derivative at $x \in [-a, a]$, the Voronovskaja-type result

$$\lim_{n\to\infty} n\left(L_n(f;x) - f(x)\right) = \frac{f^{(2)}(x)}{6} \tag{5}$$

is valid (see [9, 3.32]).

In [11, 12] Ivan and Raşa gave a more refined analysis of the convergence behaviour of the operators L_n. They obtained the asymptotic relation

$$L_n(f;x) = f(x) + \frac{f^{(2)}(x)}{6n} + \frac{f^{(4)}(x)}{72n^2} + \frac{5f^{(6)}(x) - 36f^{(4)}(x)}{6480n^3} + o\left(n^{-3}\right)$$

as $n \to \infty$.

In [6] the complete asymptotic expansion for the operators L_n in the case of equidistant knots is obtained in terms of the central factorial numbers of first and second kind.

The purpose of this paper is to derive the complete asymptotic expansion for the operators L_n when the condition of equidistance of the knots is dropped out. We obtain the expansion

$$L_n(f;x) \sim f(x) + \sum_{j=1}^{\infty} \frac{c_j(f;x)}{(n+1)^{\overline{j}}} \qquad (n \to \infty), \tag{6}$$

provided $f \in C[-a-1, a+1]$ admits derivatives of sufficiently high order at $x \in [-a, a]$. By $x^{\overline{m}} = x(x+1)\dots(x+m-1)$, $x^{\overline{0}} = 1$, we denote the rising factorial and $x^{\underline{m}} = x(x-1)\dots(x-m+1)$, $x^{\underline{0}} = 1$ denotes the falling factorial.

Formula (6) means that, for all $m = 1, 2, \dots$, there holds

$$L_n(f;x) = f(x) + \sum_{j=1}^{m} \frac{c_j(f;x)}{(n+1)^{\overline{j}}} + o(n^{-m}) \qquad (n \to \infty).$$

In the special case $m = 1$ one obtains the Voronovskaja-type result (5) with $c_1(f;x) = f^{(2)}(x)/6$. All coefficients $c_j(f;x)$ are calculated explicitly in terms of exponential Bell polynomials. Recall that the (exponential) partial Bell polynomials are the polynomials $\mathbf{B}_{n,k} = \mathbf{B}_{n,k}(x_1, x_2, \dots)$ in an infinite number of variables $x_1, x_2, \dots$, defined by the formal double series expansion:

$$\exp\left(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = \sum_{n,k\geq 0} \mathbf{B}_{n,k} \frac{t^n}{n!} u^k \quad [7, \text{p.133,[3a]}].$$

The (exponential) complete Bell polynomials $\mathbf{Y}_r(x_1, x_2, \ldots)$ are defined by

$$\exp\left(\sum_{m\geq 1} x_m \frac{z^m}{m!}\right) = 1 + \sum_{r\geq 1} \mathbf{Y}_r(x_1, x_2, \ldots)\frac{z^r}{r!}. \tag{7}$$

In particular, $\mathbf{Y}_0 = 1$. Properties of Bell polynomials can be found in [7, pp. 133–137].

We mention that analogous results for the Bernstein-Kantorovich operators, the Meyer-König and Zeller operators and the operators of Butzer, Bleimann and Hahn can be found in [4, 1, 3, 2, 5].

3. A representation for divided differences of monomials

Throughout this section we put

$$s_m = \sum_{k=0}^{n} z_k^m \qquad (m = 1, 2, \ldots).$$

We shall prove the following representation formula for $[z_0, \ldots, z_n; e_{n+r}]$.

THEOREM 3.1. *Let $r \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $z_0, \ldots, z_n \in \mathbb{C}$. Then, we have the representation*

$$[z_0, \ldots, z_n; e_{n+r}] = \frac{1}{r!}\mathbf{Y}_r(0!s_1, 1!s_2, \ldots).$$

PROOF. Let g be a function analytic in a simply connected region D. Let C be a simple, closed, rectifiable curve that lies in D and contains the points $z_0, \ldots, z_n$ in its interior. Using the Cauchy integral formula and Eq. (1) we obtain the well-known formula

$$[z_0, \ldots, z_n; g] = \frac{1}{2\pi i}\oint_C \frac{g(z)}{(z - z_0)\ldots(z - z_n)}\, dz$$

(see, e.g., [8, p. 67]). For $R > \max_{0\leq j\leq n} |z_j|$, there holds

$$\begin{aligned}[z_0, \ldots, z_n; e_{n+r}] &= \tfrac{1}{2\pi i}\oint_{|z|=R} \tfrac{z^{r-1}}{(1-\frac{z_0}{z})\ldots(1-\frac{z_n}{z})}\, dz \\ &= \tfrac{1}{2\pi i}\oint_{|z|=R^{-1}} \tfrac{z^{-r-1}}{(1-z_0 z)\ldots(1-z_n z)}\, dz. \end{aligned} \tag{8}$$

Since $|z_j z| < 1$ on the curve $|z| = R^{-1}$, we have

$$\begin{aligned}&\log\frac{1}{(1-z_0z)\dots(1-z_nz)}\\ =\;&-\sum_{j=0}^{n}\log\left(1-z_jz\right)=\sum_{j=0}^{n}\sum_{k=1}^{\infty}\frac{1}{k}\left(z_jz\right)^k=\sum_{k=1}^{\infty}\frac{s_k}{k}z^k.\end{aligned}$$

Note that the logarithm is univalent because $[(1-z_0z)\dots(1-z_nz)]^{-1}$ is contained in the right half-plane for $|z|=R^{-1}$, if R is sufficiently large. Thus, application of Eq. (7) implies

$$\begin{aligned}&[(1-z_0z)\dots(1-z_nz)]^{-1}\\ =\;&\exp\left(\sum_{m\geq1}(m-1)!s_m\frac{z^m}{m!}\right)=1+\sum_{k\geq1}\mathbf{Y}_k\left(0!s_1,1!s_2,\dots\right)\frac{z^k}{k!}.\end{aligned}$$

The latter power series expansion is convergent for $|z|\leq R^{-1}$. Inserting it into Eq. (8) yields

$$\begin{aligned}&[z_0,\dots,z_n;e_{n+r}]\\ =\;&\frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{Y}_k\left(0!s_1,1!s_2,\dots\right)\oint\limits_{|z|=R^{-1}}z^{k-r-1}\,dz\\ =\;&\frac{1}{r!}\mathbf{Y}_r\left(0!s_1,1!s_2,\dots\right).\end{aligned}$$

■

For the convenience of the reader we explicitly calculate the exact expressions of $[z_0,\dots,z_n;e_{n+r}]$ for $r=0,\dots,5$.

$$\begin{aligned}[z_0,\dots,z_n;e_{n+1}]&=s_1\\ [z_0,\dots,z_n;e_{n+2}]&=\left(s_2+s_1^2\right)/2\\ [z_0,\dots,z_n;e_{n+3}]&=\left(2s_3+3s_2s_1+s_1^3\right)/6\\ [z_0,\dots,z_n;e_{n+4}]&=\left(6s_4+8s_3s_1+3s_2^2+6s_2s_1^2+s_1^4\right)/24\\ [z_0,\dots,z_n;e_{n+5}]&=(24s_5+30s_4s_1+20s_3s_2+20s_3s_1^2\\ &\quad+15s_2^2s_1+10s_2s_1^3+s_1^5)/120\end{aligned}$$

Further formulae for $[z_0,\dots,z_n;e_{n+r}]$ can be constructed by using the well-known formula

$$\mathbf{Y}_r\left(x_1,x_2,\dots\right)=\sum_{k=0}^{r}\mathbf{B}_{r,k}\left(x_1,x_2,\dots\right)\tag{9}$$

(see, e.g., [7, Formula (3c), p. 134]). We note that the partial exponential Bell polynomials $\mathbf{B}_{n,k}(x_1, x_2, \ldots)$ are listed in [7, p. 307–308] for all $n, k \leq 12$.

4. An asymptotic expansion for divided differences of monomials

In this section we obtain a complete asymptotic expansion for $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}]$ as $n \to \infty$, for all $r \in \mathbb{N}_0$.

Throughout this paper we put

$$S_{n,m} = \frac{1}{n+1} \sum_{k=0}^{n} z_{n,k}^m \qquad (m, n = 0, 1, \ldots). \tag{10}$$

In [12] Ivan and Raşa proved the following result.

THEOREM 4.1. *Assume that the triangular matrix of real knots* $(x_{n,k})$ *satisfies the conditions*

$$-1 \leq x_{n,0} < x_{n,1} < \cdots < x_{n,n} \leq 1 \qquad (n = 0, 1, \ldots)$$

and

$$x_{n,n-k} = -x_{n,k} \qquad (k = 0, \ldots, n;\ n = 0, 1, \ldots). \tag{11}$$

If the limit $\lim_{n\to\infty} S_{n,2} =: 2\lambda$ *exists, then, for all* $k = 0, 1, \ldots$, *we have the asymptotic relation*

$$\lim_{n\to\infty} n^{-k} [x_{n,0}, \ldots, x_{n,n}; e_{n+2k}] = \frac{\lambda^k}{k!}.$$

We will generalize the above theorem in three directions. Firstly, we consider a fairly general system of complex knots $(z_{n,k})$. Secondly, we deal with $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}]$, for all $r \in \mathbb{N}_0$.

The next theorem presents an explicit expression for $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}]$ revealing its asymptotic behaviour as n tends to infinity ($r = 0, 1, \ldots$).

THEOREM 4.2. *Let* $(z_{n,k})$ *be a triangular matrix of complex knots* $(z_{n,k})$ *such that, for all* $n = 0, 1, \ldots$, *the numbers* $z_{n,0}, \ldots, z_{n,n}$ *are pairwise different. Suppose that*

$$S_{n,1} = 0 \qquad (n = 0, 1, \ldots) \tag{12}$$

and

$$S_{n,m} = O(1) \quad (n \to \infty) \qquad \text{for all } m = 2, 3, \ldots, \tag{13}$$

where $S_{n,m}$ is as defined in (10). *Then, for all $r = 0, 1, \ldots$, we have the asymptotic relation*

$$[z_{n,0}, \ldots, z_{n,n}; e_{n+r}] = \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{(n+1)^k}{(r-k)!} \mathbf{B}_{r-k,k}\left(\frac{1!}{2}S_{n,2}, \frac{2!}{3}S_{n,3}, \ldots\right). \tag{14}$$

REMARK 4.3. *Note that condition* (13) *guarantees*

$$\mathbf{B}_{r-k,k}\left(\frac{1!}{2}S_{n,2}, \frac{2!}{3}S_{n,3}, \ldots\right) = O(1) \quad (n \to \infty).$$

Therefore, Eq. (14) *yields* $[z_{n,0}, \ldots, z_{n,n}; e_{n+r}] = O\left(n^{\lfloor r/2 \rfloor}\right)$ *as* $n \to \infty$.

Proof of Theorem 4.2. By Theorem 3.1, Eq. (9) and the fact that the partial exponential Bell polynomial $\mathbf{B}_{r,k}$ is homogeneous of degree k, we obtain the representation

$$[z_{n,0}, \ldots, z_{n,n}; e_{n+r}] = \frac{1}{r!}\sum_{k=0}^{r}(n+1)^k \mathbf{B}_{r,k}(0, 1!S_{n,2}, 2!S_{n,3}, \ldots).$$

Using properties of the partial exponential Bell polynomials (see, e.g., [7, Formula (3l'), p. 136]) it follows Eq. (14). ■

The following corollary contains more explicit formulae which follow from the exact expression

$$\mathbf{B}_{r,k}(x_1, x_2, \ldots) = r! \sum \prod_{j=1}^{r} \frac{x_j^{\nu_j}}{\nu_j!\,(j!)^{\nu_j}},$$

where the summation takes place over all integers $\nu_1, \nu_2, \ldots \geq 0$, such that $\nu_1 + 2\nu_2 + 3\nu_3 + \cdots = r$ and $\nu_1 + \nu_2 + \nu_3 + \cdots = k$ (see, e.g., [7, Theorem A,p. 134]).

COROLLARY 4.4. *Under the conditions of* Theorem 4.2, *we have, for $r = 4, 5, \ldots$, the asymptotic relations*

$$\begin{aligned}&(n+1)^{-r}[z_{n,0}, \ldots, z_{n,n}; e_{n+2r}] \\ &= \frac{S_{n,2}^r}{2^r r!} + \left(\frac{S_{n,2}^{r-2}S_{n,4}}{2^r\,(r-2)!} + \frac{S_{n,2}^{r-3}S_{n,3}^2}{9\cdot 2^{r-2}\,(r-3)!}\right)(n+1)^{-1} + O\left(n^{-2}\right),\end{aligned}$$

and

$$(n+1)^{-r}[z_{n,0}, \ldots, z_{n,n}; e_{n+2r+1}]$$

$$= \frac{S_{n,2}^{r-1} S_{n,3}}{3 \cdot 2^{r-1} (r-1)!} + \left(\frac{S_{n,2}^{r-2} S_{n,5}}{5 \cdot 2^{r-2} (r-2)!} + \frac{S_{n,2}^{r-3} S_{n,3} S_{n,4}}{3 \cdot 2^{r-1} (r-3)!} + \frac{S_{n,2}^{r-4} S_{n,3}^3}{3^4 \cdot 2^{r-3} (r-4)!} \right) (n+1)^{-1} + O\left(n^{-2}\right)$$

as $n \to \infty$.

REMARK 4.5. *Both formulae of the corollary are valid also for* $r = 0, 1, 2, 3$ *if the quantities* $S_{n,m}^k$ *are interpreted to be* 0 *if* $k < 0$. *More precisely, we have*

$$[z_{n,0}, \ldots, z_{n,n}; e_{n+r}] = \begin{cases} 1 & (r=0), \\ 0 & (r=1), \\ S_{n,2}(n+1)/2 & (r=2), \\ S_{n,3}(n+1)/3 & (r=3), \\ S_{n,2}^2 (n+1)^2/8 + S_{n,4}(n+1)/4 & (r=4), \\ S_{n,2} S_{n,3} (n+1)^2/6 + S_{n,5}(n+1)/5 & (r=5). \end{cases}$$

REMARK 4.6. *The result of Ivan and Raşa* (Theorem 4.1) *follows from* Corollary 4.4 *since condition* (11) *immediately implies* (12). *Moreover,* (13) *is valid if all knots satisfy* $|z_{n,k}| \leq 1$.

5. Application to a positive linear operator

Let

$$-1 \leq z_{n,0} < z_{n,1} < \cdots < z_{n,n} \leq 1 \qquad (n = 0, 1, \ldots).$$

Obviously, we then have $|S_{n,m}| \leq 1$, for all $n, m = 0, 1, \ldots$, (cf. Remark 4.6).

Let $a > 0$ be a real number. For $n = 1, 2, \ldots$, let the operators $L_n : C[-a-1, a+1] \to C[-a, a]$ be given by

$$L_n(f; x) = n![x + z_{n,0}, \ldots, x + z_{n,n}; f^{(-n)}] \qquad (-a \leq x \leq a), \quad (15)$$

where $f^{(-n)}$ is an n-th antiderivative of f. We derive a complete asymptotic expansion for the operators L_n as $n \to \infty$.

For $q = 1, 2, \ldots$, and fixed $x \in [-a, a]$, we define $K^{[q]}(x)$ to be the class of all functions $f \in C[-a-1, a+1]$ which are q-times differentiable at x.

In the following proposition we derive an asymptotic expression for $L_n(f; x)$ as $n \to \infty$.

PROPOSITION 5.1. *Let $q \in \mathbb{N}$ and $x \in [-a, a]$. Suppose that $S_{n,1} = 0$ $(n = 0, 1, \ldots)$. Then, for $f \in K^{[2q]}(x)$, the operators L_n satisfy the asymptotic relation*

$$\begin{aligned} & L_n(f;x) \\ = \; & f(x) + \sum_{r=1}^{2q} \frac{f^{(r)}(x)}{(n+1)^{\overline{r}}} \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{(n+1)^k}{(r-k)!} \mathbf{B}_{r-k,k}\left(\frac{1!}{2} S_{n,2}, \frac{2!}{3} S_{n,3}, \ldots\right) \\ & + o\left(n^{-q}\right) \end{aligned}$$

as $n \to \infty$, where $S_{n,m}$ is as defined in Eq. (10).

For the convenience of the reader we list an explicit expression approximating $L_n(f;x)$ of order $o\left(n^{-3}\right)$:

$$\begin{aligned} & L_n(f;x) \\ = \; & f(x) + \frac{S_{n,2}}{2(n+2)} f^{(2)}(x) + \frac{S_{n,3}}{3(n+2)(n+3)} f^{(3)}(x) \\ & + \frac{(n-2) S_{n,2}^2 + 2 S_{n,4}}{48(n+1)(n+2)(n+3)} f^{(4)}(x) + \frac{S_{n,2} S_{n,3}}{6(n+1)(n+2)(n+3)} f^{(5)}(x) \\ & + \frac{S_{n,2}^3}{48(n+1)(n+2)(n+3)} f^{(6)}(x) + o(n^{-3}) \qquad (n \to \infty). \end{aligned}$$

In the following theorem we present the complete asymptotic expansion for the operators L_n as a reciprocal factorial series.

THEOREM 5.2. (Complete asymptotic expansion for the operators L_n). *Let $q \in \mathbb{N}$ and $x \in [-a, a]$. Suppose that*

$$S_{n,1} = 0 \qquad (n = 0, 1, \ldots),$$

where $S_{n,m}$ is as defined in (10). *Then, for $f \in K^{[2q]}(x)$, the operators L_n possess the asymptotic expansion*

$$L_n(f;x) = f(x) + \sum_{j=1}^{q} \frac{c_j(f;x)}{(n+1)^{\overline{j}}} + o\left(n^{-q}\right)$$

as $n \to \infty$, where

$$\begin{aligned} c_j(f;x) \; = \; & \sum_{r=j}^{2j} f^{(r)}(x) \sum_{k=\lfloor (r+1)/2 \rfloor}^{j} \frac{1}{k!} \mathbf{B}_{r-k,k}\left(\frac{1!}{2} S_{n,2}, \frac{2!}{3} S_{n,3}, \ldots\right) \\ & \times \sum_{i=0}^{j-k} \binom{r-k}{i} \sigma_{r-k-i}^{r-j} (1-r)^i \end{aligned}$$

and σ_j^i denote the Stirling numbers of the second kind.

Recall that the Stirling numbers of the second kind are defined by the equations

$$x^j = \sum_{i=0}^{j} \sigma_j^i x^{\underline{i}} \qquad (j = 0, 1, \ldots). \tag{16}$$

For the convenience of the reader we list the explicit expressions for the initial coefficients:

$$\begin{aligned}
c_0(f;x) &= f(x)\\
c_1(f;x) &= \frac{1}{2} S_{n,2} f^{(2)}(x)\\
c_2(f;x) &= \frac{1}{24}\left(-12 S_{n,2} f^{(2)}(x) + 8 S_{n,3} f^{(3)}(x) + 3 S_{n,2}^2 f^{(4)}(x)\right)\\
c_3(f;x) &= \frac{1}{48}\Big(- 32 S_{n,3} f^{(3)}(x) + \left(-30 S_{n,2}^2 + 12 S_{n,4}\right) f^{(4)}(x)\\
&\quad + 8 S_{n,2} S_{n,3} f^{(5)}(x) + S_{n,2}^3 f^{(6)}(x)\Big)
\end{aligned}$$

In the proof we will use a general approximation theorem for positive linear operators due to Sikkema [24, Theorem 1 and 2].

LEMMA 5.3 (Sikkema). *For $q \in \mathbb{N}$ and fixed $x \in [-a, a]$, let $A_n : K^{[2q]}(x) \to C[-a, a]$ be a sequence of positive linear operators. If, for $s = 0, \ldots, 2q+2$,*

$$A_n\left((\cdot - x)^{2s}; x\right) = O\left(n^{-\lfloor (s+1)/2 \rfloor}\right) \qquad (n \to \infty),$$

then we have, for each $f \in K^{[2q]}(x)$,

$$A_n(f;x) = \sum_{s=0}^{2q} \frac{f^{(s)}(x)}{s!} A_n\left((\cdot - x)^s; x\right) + o\left(n^{-q}\right) \qquad (n \to \infty). \tag{17}$$

Furthermore, if $f \in K^{[2q+2]}(x)$, the term $o(n^{-q})$ in Eq. (17) *can be replaced by $O\left(n^{-(q+1)}\right)$.*

Proof of Proposition 5.1. By the definition of the operators L_n we obtain for their central moments the representation

$$\begin{aligned}
L_n((\cdot - x)^r; x) &= \frac{n! r!}{(n+r)!} [x + z_{n,0}, \ldots, x + z_{n,n}; (\cdot - x)^{n+r}]\\
&= \frac{n! r!}{(n+r)!} [z_{n,0}, \ldots, z_{n,n}; e_{n+r}]
\end{aligned}$$

and Theorem 4.2 yields

$$L_n((\cdot - x)^r; x) = \frac{n!r!}{(n+r)!} \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{(n+1)^k}{(r-k)!} \mathbf{B}_{r-k,k}\left(\frac{1!}{2}S_{n,2}, \frac{2!}{3}S_{n,3}, \ldots\right)$$

which implies $L_n((\cdot - x)^r; x) = O\left(n^{-\lfloor (r+1)/2 \rfloor}\right)$ $(n \to \infty;\ r = 0, 1, \ldots)$. Therefore, we can apply Lemma 5.3 which completes the proof of the proposition. ■

Proof of Theorem 5.2. By Proposition 5.1, we have

$$\begin{aligned} & L_n(f; x) \\ = & \sum_{r=0}^{2q} \frac{f^{(r)}(x)}{(n+1)^{\overline{r}}} \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{(n+1)^k}{(r-k)!} \mathbf{B}_{r-k,k}\left(\frac{1!}{2}S_{n,2}, \frac{2!}{3}S_{n,3}, \ldots\right) \qquad (18) \\ & + o\left(n^{-q}\right) \qquad (n \to \infty). \end{aligned}$$

Application of Eq. (16) yields

$$\begin{aligned} (n+1)^k &= \sum_{i=0}^{k} \binom{k}{i} (1-r)^{k-i} (n+r)^i \\ &= \sum_{j=0}^{k} (n+r)^{\underline{j}} \sum_{i=j}^{k} \binom{k}{i} \sigma_i^j (1-r)^{k-i}. \end{aligned}$$

Inserting this into (19) and using

$$\frac{(n+r)^{\underline{j}}}{(n+1)^{\overline{r}}} = \frac{1}{(n+1)^{\overline{r-j}}}$$

yields after some manipulations the assertion of Theorem 5.2. ■

We close this section by considering special schemes of knots. In the case of Chebyshev's knots

$$z_{n,k} = \cos\left(\frac{2k+1}{2(n+1)}\pi\right) \qquad (k = 0, \ldots, n;\ n = 0, 1, \ldots)$$

we have

$$S_{n,2m} = \frac{1}{4^m}\binom{2m}{m}, \qquad S_{n,2m+1} = 0 \qquad (m, n = 0, 1, \ldots).$$

Thus, we obtain the asymptotic expansion

$$\begin{aligned} L_n(f; x) &= f(x) + \frac{f^{(2)}(x)}{4(n+2)} + \frac{f^{(4)}(x)}{32(n+2)(n+3)} \\ &\quad + \frac{f^{(6)}(x)}{384(n+2)(n+3)(n+4)} + o\left(n^{-3}\right) \qquad (n \to \infty). \end{aligned}$$

In the case of equidistant knots

$$z_{n,k} = -1 + \frac{2k}{n} \qquad (k = 0, \ldots, n;\ n = 0, 1, \ldots)$$

we have

$$S_{n,2} = \frac{n+2}{3n}, \qquad S_{n,4} = \frac{n+2}{15n^3}\left(3n^2 + 6n - 4\right), \qquad S_{n,2m+1} = 0,$$

$(m, n = 0, 1, \ldots)$. Thus, we obtain after simple calculations the asymptotic relation

$$\begin{aligned} L_n(f;x) \;=\; & f(x) + \frac{f^{(2)}(x)}{6n} + \frac{(60 + 80n + 33n^2 + 5n^3)}{360n^2\,(n+1)\,(n+2)\,(n+3)} f^{(4)}(x) \\ & + \frac{f^{(6)}(x)}{1296(n+1)(n+2)(n+3)} + o\left(n^{-3}\right) \qquad (n \to \infty), \end{aligned}$$

giving back the result of Ivan and Raşa [11] (see also [6]).

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On Ostrowski Like Integral Inequality for the Čebyšev Difference and Applications

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Abstract. Some integral inequalities similar to the Ostrowski's result for Čebyšev's difference and applications for perturbed generalized Taylor's formula are given.

Key Words: Ostrowski's inequality, Čebyšev's difference, Taylor's formula.

AMS Subj. Class.: Primary 26D15; Secondary 26D10

1. Introduction

In [**?**], A. Ostrowski proved the following inequality of Grüss type for the difference between the integral mean of the product and the product of the integral means, or *Čebyšev's difference*, for short:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{8}(b-a)(M-m)\,\|f'\|_{[a,b],\infty} \tag{1.1}$$

provided g is measurable and satisfies the condition

$$-\infty < m \leq g(x) \leq M < \infty \quad \text{for a.e. } x \in [a,b]\,; \tag{1.2}$$

and f is absolutely continuous on $[a,b]$ with $f' \in L_\infty[a,b]$.

The constant $\frac{1}{8}$ is best possible in (**??**) in the sense that it cannot be replaced by a smaller constant.

In this paper we establish some similar results. Applications for perturbed generalized Taylor's formulae are also provided.

2. Integral Inequalities

The following result holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be an absolutely continuous function with $f'\in L_\infty[a,b]$ and $g\in L_1[a,b]$. Then one has the inequality*

$$\begin{aligned}(2.1)\quad &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ &\qquad\le\|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx.\end{aligned}$$

*The inequality (**??**) is sharp in the sense that the constant $c=1$ in the left hand side cannot be replaced by a smaller one.*

Proof. We observe, by simple computation, that one has the identity

$$\begin{aligned}(2.2)\quad T(f,g)&:=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\\ &=\frac{1}{b-a}\int_a^b\left[f(x)-f\left(\frac{a+b}{2}\right)\right]\left[g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right]dx.\end{aligned}$$

Since f is absolutely continuous, we have

$$\int_{\frac{a+b}{2}}^x f'(t)\,dt=f(x)-f\left(\frac{a+b}{2}\right)$$

and thus, the following identity that is in itself of interest,

$$(2.3)\qquad T(f,g)=\frac{1}{b-a}\int_a^b\left(\int_{\frac{a+b}{2}}^x f'(t)\,dt\right)\left[g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right]dx$$

holds.

Since

$$\left|\int_{\frac{a+b}{2}}^x f'(t)\,dt\right|\le\left|x-\frac{a+b}{2}\right|ess\sup_{\substack{t\in\left[x,\frac{a+b}{2}\right]\\(t\in\left[\frac{a+b}{2},x\right])}}|f'(t)|=\left|x-\frac{a+b}{2}\right|\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}$$

for any $x\in[a,b]$, then taking the modulus in (**??**), we deduce

$$\begin{aligned}|T(f,g)|&\le\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\\ &\le\sup_{x\in[a,b]}\left\{\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}\right\}\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\\ &=\max\left\{\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty},\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right\}\\ &\qquad\times\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\\ &=\|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\end{aligned}$$

and the inequality (**??**) is proved.

To prove the sharpness of the constant $c = 1$, assume that (**??**) holds with a positive constant $D > 0$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq D\,\|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx. \tag{2.4}$$

If we choose $\mathbb{K} = \mathbb{R}$, $f(x) = x - \frac{a+b}{2}$, $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$,

$$g(x) = \begin{cases} -1 & \text{if} \quad x \in \left[a, \frac{a+b}{2}\right] \\ 1 & \text{if} \quad x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \\ = \;&\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|dx = \frac{b-a}{4}, \end{aligned}$$

$$\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx = \frac{b-a}{4},$$

$$\|f'\|_{[a,b],\infty} = 1$$

and by (**??**) we deduce

$$\frac{b-a}{4} \leq D\cdot\frac{b-a}{4},$$

giving $D \geq 1$, and the sharpness of the constant is proved. □

The following corollary may be useful in practice.

Corollary 1. *Let $f : [a,b] \to \mathbb{K}$ be an absolutely continuous function on $[a,b]$ with $f' \in L_\infty[a,b]$. If $g \in L_\infty[a,b]$, then one has the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(b-a)\,\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}. \tag{2.5}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Obviously,

$$\begin{aligned}(2.6)\qquad &\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\\ &\leq\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|dx\\ &=\frac{b-a}{4}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}.\end{aligned}$$

Using (**??**) and (**??**) we deduce (**??**).

Assume that (**??**) holds with a constant $E>0$ instead of $\frac{1}{4}$, i.e.,

$$\begin{aligned}(2.7)\qquad &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ &\qquad\leq E(b-a)\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}.\end{aligned}$$

If we choose the same functions as in Theorem **??**, then we get from (**??**)

$$\frac{b-a}{4}\leq E(b-a),$$

giving $E\geq\frac{1}{4}$. □

Corollary 2. *Let f be as in Theorem* **??**. *If $g\in L_p[a,b]$ where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then one has the inequality:*

$$\begin{aligned}(2.8)\qquad &\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|\\ \leq\;&\frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],p}.\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. By Hölder's inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, one has

$$\begin{aligned}(2.9)\qquad &\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx\\ \leq\;&\frac{1}{b-a}\left(\int_a^b\left|x-\frac{a+b}{2}\right|^q dx\right)^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}}\\ =\;&\frac{1}{b-a}\left[\frac{(b-a)^{q+1}}{2^q(q+1)}\right]^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}}\\ =\;&\frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}}.\end{aligned}$$

Using (**??**) and (**??**), we deduce (**??**).

Now, if we assume that the inequality (**??**) holds with a constant $F > 0$ instead of $\frac{1}{2}$ and choose the same functions f and g as in Theorem **??**, we deduce

$$\frac{b-a}{4} \leq \frac{F}{(q+1)^{\frac{1}{q}}} (b-a), \ q > 1$$

giving $F \geq \frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q > 1$. Letting $q \to 1+$, we deduce $F \geq \frac{1}{2}$, and the corollary is proved. □

Finally, we also have

Corollary 3. *Let f be as in Theorem* **??**. *If $g \in L_1[a,b]$, then one has the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{2} \|f'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y) dy \right\|_{[a,b],1}. \tag{2.10}$$

Proof. Since

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| \left| g(x) - \frac{1}{b-a} \int_a^b g(y) dy \right| dx \\ \leq & \sup_{x \in [a,b]} \left| x - \frac{a+b}{2} \right| \left\| g - \frac{1}{b-a} \int_a^b g(y) dy \right\|_{[a,b],1} \\ = & \frac{b-a}{2} \left\| g - \frac{1}{b-a} \int_a^b g(y) dy \right\|_{[a,b],1} \end{aligned}$$

the inequality (**??**) follows by (**??**). □

Remark 1. *Similar inequalities may be stated for weighted integrals. These inequalities and their applications in connection to Schwartz's inequality will be considered in* [**?**].

3. Applications to Taylor's Formula

In the recent paper [**?**], M. Matić, J. E. Pečarić and N. Ujević proved the following generalized Taylor formula.

Theorem 2. *Let $\{P_n\}_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is, $P_n'(t) = P_{n-1}(t)$ for $n \geq 1$, $n \in \mathbb{N}$, $P_0(t) = 1$, $t \in \mathbb{R}$. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is a function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then*

$$f(x) = \tilde{T}_n(f; a, x) + \tilde{R}_n(f; a, x), \ \ x \in I, \tag{3.1}$$

where

$$\tilde{T}_n(f; a, x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] \tag{3.2}$$

and

$$\tilde{R}_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt. \tag{3.3}$$

For some particular instances of harmonic sequences, they obtained the following Taylor-like expansions:

$$f(x) = T_n^{(M)}(f;a,x) + R_n^{(M)}(f;a,x), \quad x \in I, \tag{3.4}$$

where

$$T_n^{(M)}(f;a,x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right], \tag{3.5}$$

$$R_n^{(M)}(f;a,x) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n f^{(n+1)}(t)\, dt; \tag{3.6}$$

and

$$f(x) = T_n^{(B)}(f;a,x) + R_n^{(B)}(f;a,x), \quad x \in I, \tag{3.7}$$

where

$$\begin{aligned} T_n^{(B)}(f;a,x) &= f(a) + \frac{x-a}{2} [f'(x) + f'(a)] \\ &\quad - \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{(x-a)^{2k}}{(2k)!} B_{2k} \left[f^{(2k)}(x) - f^{(2k)}(a) \right], \end{aligned} \tag{3.8}$$

and $[r]$ is the integer part of r. Here, B_{2k} are the Bernoulli numbers, and

$$R_n^{(B)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x B_n \left(\frac{t-a}{x-a} \right) f^{(n+1)}(t)\, dt, \tag{3.9}$$

where $B_n(\cdot)$ are the Bernoulli polynomials, respectively.

In addition, they proved that

$$f(x) = T_n^{(E)}(f;a,x) + R_n^{(E)}(f;a,x), \quad x \in I, \tag{3.10}$$

where

$$\begin{aligned} &T_n^{(E)}(f;a,x) \\ &= f(a) + 2 \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(x-a)^{2k-1} \left(4^k - 1\right)}{(2k)!} B_{2k} \left[f^{(2k-1)}(x) + f^{(2k-1)}(a) \right] \end{aligned} \tag{3.11}$$

and

$$R_n^{(E)}(f;a,x) = (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n \left(\frac{t-a}{x-a} \right) f^{(n+1)}(t)\, dt, \tag{3.12}$$

where $E_n(\cdot)$ are the Euler polynomials.

In [**?**], S.S. Dragomir was the first author to introduce the perturbed Taylor formula

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!} \left[f^{(n)}; a, x \right] + G_n(f;a,x), \tag{3.13}$$

where

$$T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{3.14}$$

and

$$\left[f^{(n)};a,x\right]:=\frac{f^{(k)}(x)-f^{(k)}(a)}{x-a};$$

and had the idea to estimate the remainder $G_n(f;a,x)$ by using Grüss and *Čebyšev* type inequalities.

In [**?**], the authors generalized and improved the results from [**?**]. We mention here the following result obtained via a pre-Grüss inequality (see [**?**, Theorem 3]).

Theorem 3. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I\subset\mathbb{R}$ be a closed interval and $a\in I$. Suppose $f:I\to\mathbb{R}$ is as in Theorem* **??**. *Then for all $x\in I$ we have the perturbed generalized Taylor formula:*

$$\begin{aligned}(3.15)\qquad f(x) &= \tilde{T}_n(f;a,x)+(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right]\left[f^{(n)};a,x\right]\\ &\quad+\tilde{G}_n(f;a,x).\end{aligned}$$

For $x\geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimate

$$(3.16)\qquad \left|\tilde{G}_n(f;a,x)\right|\leq\frac{x-a}{2}\sqrt{T(P_n,P_n)}\left[\Gamma(x)-\gamma(x)\right],$$

provided that $f^{(n+1)}$ is bounded and

$$(3.17)\qquad \Gamma(x):=\sup_{t\in[a,x]}f^{(n+1)}(t)<\infty,\quad \gamma(x):=\inf_{t\in[a,x]}f^{(n+1)}(t)>-\infty,$$

where $T(\cdot,\cdot)$ is the Čebyšev functional on the interval $[a,x]$, that is, we recall

$$(3.18)\qquad T(g,h):=\frac{1}{x-a}\int_a^x g(t)h(t)\,dt-\frac{1}{x-a}\int_a^x g(t)\,dt\cdot\frac{1}{x-a}\int_a^x h(t)\,dt.$$

In [**?**], the author has proved the following result improving the estimate (**??**).

Theorem 4. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f:I\to\mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)}\in L_2(I)$. If $x\geq a$, then we have the inequality*

$$\begin{aligned}(3.19)\qquad &\left|\tilde{G}_n(f;a,x)\right|\\ &\leq (x-a)\left[T(P_n,P_n)\right]^{\frac{1}{2}}\left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2-\left(\left[f^{(n)};a,x\right]\right)^2\right]^{\frac{1}{2}}\\ &\quad\left(\leq\frac{x-a}{2}\left[T(P_n,P_n)\right]^{\frac{1}{2}}\left[\Gamma(x)-\gamma(x)\right],\quad \text{if}\ \ f^{(n+1)}\in L_\infty[a,x]\right),\end{aligned}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $[a,x]$, i.e.,

$$\left\|f^{(n+1)}\right\|_2=\left(\int_a^x\left|f^{(n+1)}(t)\right|^2dt\right)^{\frac{1}{2}}.$$

Remark 2. *If $f^{(n+1)}$ is unbounded on (a,x) but $f^{(n+1)}\in L_2(a,x)$, then the first inequality in (**??**) can still be applied, but not the Matić-Pečarić-Ujević result (**??**) which requires the boundedness of the derivative $f^{(n+1)}$.*

The following corollary [**?**] improves Corollary 3 of [**?**], which deals with the estimation of the remainder for the particular perturbed Taylor-like formulae (**??**), (**??**) and (**??**).

Corollary 4. *With the assumptions in Theorem* **??**, *we have the following inequalities*

$$\left|\tilde{G}_n^{(M)}(f;a,x)\right| \leq \frac{(x-a)^{n+1}}{n!2^n\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.20}$$

$$\left|\tilde{G}_n^{(B)}(f;a,x)\right| \leq (x-a)^{n+1}\left[\frac{|B_{2n}|}{(2n)!}\right]^{\frac{1}{2}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.21}$$

$$\begin{aligned}\left|\tilde{G}_n^{(E)}(f;a,x)\right| &\\ \leq 2(x-a)^{n+1}&\left[\frac{\left(4^{n+1}-1\right)|B_{2n+2}|}{(2n+2)!}-\left[\frac{2\left(2^{n+2}-1\right)B_{n+2}}{(n+1)!}\right]^2\right]^{\frac{1}{2}}\\ &\times\sigma\left(f^{(n+1)};a,x\right),\end{aligned} \tag{3.22}$$

and

$$|G_n(f;a,x)| \leq \frac{n(x-a)^{n+1}}{(n+1)!\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.23}$$

where, as in [**?**],

$$\tilde{G}_n^{(M)}(f;a,x) = f(x) - T_n^M(f;a,x) - \frac{(x-a)^{n+1}\left[1+(-1)^n\right]}{(n+1)!2^{n+1}}\left[f^{(n)};a,x\right];$$

$$\tilde{G}_n^{(B)}(f;a,x) = f(x) - T_n^B(f;a,x);$$

$$\tilde{G}_n^{(E)}(f;a,x) = f(x) - \frac{4(-1)^n(x-a)^{n+1}\left(2^{n+2}-1\right)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right],$$

$G_n(f;a,x)$ *is as defined by* (**??**),

$$\sigma\left(f^{(n+1)};a,x\right) := \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n+1)};a,x\right]\right)^2\right]^{\frac{1}{2}}, \tag{3.24}$$

and $x \geq a$, $f^{(n+1)} \in L_2[a,x]$.

Note that for all the examples considered in [**?**] and [**?**] for f, the quantity $\sigma\left(f^{(n+1)};a,x\right)$ can be completely computed and then those particular inequalities may be improved accordingly. We omit the details.

Now, observe that (for $x > a$)

$$\tilde{G}_n(f;a,x) = (-1)^n(x-a)\,T_n\left(P_n, f^{(n+1)};a,x\right),$$

where $T_n(\cdot,\cdot;a,x)$ is the Čebyšev's functional on $[a,x]$, i.e.,

$$\begin{aligned}T_n\left(P_n,f^{(n+1)};a,x\right) &= \frac{1}{x-a}\int_a^x P_n(t)\,f^{(n+1)}(t)\,dt\\ &\quad -\frac{1}{x-a}\int_a^x P_n(t)\,dt\cdot\frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt\\ &= \frac{1}{x-a}\int_a^x P_n(t)\,f^{(n+1)}(t)\,dt - [P_{n+1};a,x]\left[f^{(n)};a,x\right].\end{aligned}$$

In what follows we will use the following lemma that summarizes some integral inequalities obtained in the previous section.

Lemma 1. *Let $h : [x, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ with $h' \in L_\infty [a, b]$. Then*

$$|T_n (h, g; a, b)| \tag{3.25}$$
$$\leq \begin{cases} \frac{1}{4} (b-a) \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],\infty} & \text{if } g \in L_\infty [a, b]; \\ \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],p} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } g \in L_p [a, b]; \\ \frac{1}{2} \|h'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y)\, dy \right\|_{[a,b],1} & \text{if } g \in L_1 [a, b]; \end{cases}$$

where

$$T_n (h, g; a, b) := \frac{1}{b-a} \int_a^b h(x)\, g(x)\, dx - \frac{1}{b-a} \int_a^b h(x)\, dx \cdot \frac{1}{b-a} \int_a^b g(x)\, dx.$$

Using the above lemma, we may obtain the following new bounds for the remainder $\tilde{G}_n (f; a, x)$ in the Taylor's perturbed formula (**??**).

Theorem 5. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$\left| \tilde{G}_n (f; a, x) \right| \tag{3.26}$$
$$\leq \begin{cases} \frac{1}{4} (x-a)^2 \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x \right] \right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty [a, x]; \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}} \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x \right] \right\|_{[a,x],p} & \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and } f^{(n+1)} \in L_p [a, x]; \\ \frac{1}{2} (x-a) \|P_{n-1}\|_{[a,x],\infty} \left\| f^{(n+1)} - \left[f^{(n)}; a, x \right] \right\|_{[a,x],1}. \end{cases}$$

The proof follows by Lemma **??** on choosing $h = P_n$, $g = f^{(n+1)}$, $b = x$.
The dual result is incorporated in the following theorem.

Theorem 6. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n+1)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$\left| \tilde{G}_n (f; a, x) \right| \tag{3.27}$$
$$\leq \begin{cases} \frac{1}{4} (x-a)^2 \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],\infty} \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}} \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],p} \\ \qquad\qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} (x-a) \left\| f^{(n+2)} \right\|_{[a,x],\infty} \|P_n - [P_{n+1}; a, x]\|_{[a,x],1}. \end{cases} \tag{3.28}$$

The proof follows by Lemma **??**.

The interested reader may obtain different particular instances of integral inequalities on choosing the harmonic polynomials mentioned at the beginning of this section. We omit the details.

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Approximated Leont'ev coefficients

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Abstract

We consider Dirichlet series on convex polygons and their rate of approximation in $AC(\overline{D})$. We show that the substitution of the respective Leont'ev coefficients by appropriate interpolating sums preserves the order of approximation up to a factor $\ln n$. The estimates are given for moduli of smoothness of arbitrary order. This extends a result of Yu. I. Mel'nik in [4].

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1 Introduction

Let D be an open convex polygon with vertices at the points $a_1, \dots, a_N$, $N \geq 3$, $\overline{D}$ its closure and $\partial D = \overline{D} \setminus D$ the boundary of D. We assume $0 \in D$.

By $AC(\overline{D})$ we denote the the space of all functions $f(z)$ holomorphic in D and continuous on $\overline{D}$ with finite norm of uniform convergence $\|f\|_{AC(\overline{D})} = \sup_{z\in\overline{D}} |f(z)| < \infty$.

Consider the quasipolynomial $L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$, where $d_k \in \mathbb{C} \setminus \{0\}$, $k = 1, \dots, N$. For the set of zeros Λ of the quasipolynomial L the following results are well known [2, Ch. 1, §2][3]:

a) The zeros $\lambda_n^{(j)}$ of L with $|\lambda_n^{(j)}| > C$ for sufficient large C have the form

$$\lambda_n^{(j)} = \widetilde{\lambda}_n^{(j)} + \delta_n^{(j)}, \tag{1}$$

where $\widetilde{\lambda}_n^{(j)} = \frac{2\pi n i}{a_{j+1}-a_j} + q_j e^{i\beta_j}$ and $|\delta_n^{(j)}| \leq e^{-an}$. Here $0 < a = \text{const.}$, $j = 1, \dots, N$, $n > n_0$ and $a_{N+1} := a_1$. The parameters β_j and q_j are given by $e^{q_j(a_{j+1}-a_j)e^{i\beta_j}} =$

$-\frac{d_j}{d_{j+1}}$, where $d_{N+1} := d_1$. Hence these zeros are simple. The set of zeros Λ can be represented in the form

$$\Lambda = \{\lambda_n\}_{n=1,\dots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\dots} \right).$$

b) There is a constant $c_2 > 0$ such that there exists a positive constant A with

$$\left| \frac{e^{\lambda_n^{(j)} z}}{L'(\lambda_n^{(j)})} - (-1)^n B_j e^{\widetilde{\lambda}_n^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right| \leq A\, e^{-c_2 n} \quad \text{for all } n > n_0.$$

Here all $B_j \neq 0$ are constant, $j = 1, \dots, N$. This inequality is true for all $z \in \overline{D}$.

For simplicity reasons we assume that all zeros of L are simple.

We can expand functions $f \in AC(\overline{D})$ with respect to the family $\mathcal{E}(\Lambda) := \{e^{\lambda z}\}_{\lambda \in \Lambda}$ into a series of complex exponentials, the so called Dirichlet series

$$f(z) \sim \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)}, \tag{2}$$

where

$$\kappa_f(\lambda) = \sum_{k=1}^{N} d_k e^{a_k \lambda} \int_{a_j}^{a_k} f(\eta) e^{-\lambda \eta}\, d\eta \tag{3}$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N} d_k (a_k - a_j) \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda\left(\frac{a_j - a_k}{2\pi}\,\theta\right)}\, d\theta \tag{4}$$

are the Leont'ev coefficients. Here, the index $j = 1, \dots, N$ is arbitrary, but fixed. Many deep results on these series are due to A. F. Leont'ev [2].

We know [1] that the partial series, weighted with the generalized Jackson kernel, approximate in the order of the modulus of continuity. The question considered in this paper is, what happens if we substitute the integration in (3) or (4) by an appropriate approximating sum. Can we choose a sum, such that the rate of approximation is preserved? This problem was first posed by Yu. I. Mel'nik in [4] and solved there for first moduli of continuity. We give positive answer to that question up to a factor $\ln n$ for moduli of arbitrary order $r \in \mathbb{N}$.

In the following section we give the rate of approximation of the series (2) weighted with the generalized Jackson kernel. Then we have a closer look on (3) and (4) and give Yu. I. Mel'nik's approach for a sum for substituting the integral, such that the order of approximation is held for first moduli. In the last section we extend this result to moduli of arbitrary order.

2 Approximation with generalized Jackson weights

To estimate the regularity of functions in $AC(\overline{D})$ we consider appropriate moduli of smoothness introduced in [6] by P. M. Tamrazov. Let $\xi \in \overline{D}$, $r \in \mathbb{N}$, $\delta > 0$ and $A > 0$. Let $U(\xi, \delta) := \{z \in \mathbb{C} : |z - \xi| \leq \delta\}$ be the closed δ-ball with center ξ. We denote by $T(\overline{D}, \xi, r, \delta, A)$ the set of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with

(i) $z_i \in \overline{D} \cap U(\xi, \delta)$ for all $i = 1, \ldots, r$, and

(ii) $|z_i - z_j| \geq A\delta$ for all $i \neq j$, $i, j = 1, \ldots, r$.

If there is no vector satisfying these conditions we define $T(\overline{D}, \xi, r, \delta, A) := \emptyset$. Nevertheless for $A = 2^{-r}$ there is a $\delta > 0$ with $T(\overline{D}, \xi, r, \delta, A) \neq \emptyset$. Let $T_1 = T(\overline{D}, \xi, r+1, \delta, 2^{-r})$. Let $L(z, f, z_1, \ldots, z_r)$ be the polynomial in z of degree at most $r-1$ which interpolates f at the points $z_1, \ldots, z_r$. The r-th modulus of f is defined by

$$\omega_r(f,t) = \omega_{r,\overline{D}}(f,t)_\infty := \sup_{0<\delta\leq t} \sup_{\xi\in\overline{D}} \sup_{\substack{\mathbf{z}\in T_1 \\ \mathbf{z}=(z_0,\ldots,z_r)}} |f(z_0) - L(z_0, f, z_1, \ldots, z_r)|. \tag{5}$$

Here the supremum over the empty set is defined as zero. To estimate this modulus we consider normal majorants φ: These are bounded non-decreasing functions $\varphi :]\,0, \infty\,[\to]\,0, \infty\,[$ such that for fixed $\sigma \geq 1$ and an exponent $\gamma \geq 0$ the following normality condition holds:

$$\varphi(t\delta) \leq \sigma t^\gamma \varphi(\delta)$$

for all $\delta > 0$, $t > 1$ [5, §1]. It is shown in [7] and [8, Thm. 1] that the modulus (5) is normal, i.e., $\omega_{r,\overline{D}}(f, t\delta)_\infty \leq C \cdot t^r \cdot \omega_{r,\overline{D}}(f,\delta)_\infty$, where $C > 0$ depends on r and the polygon D only. With normal majorants we thus can define classes of regularity. By $AH_r^\varphi(\overline{D})$ we denote the class of all functions $f \in AC(\overline{D})$ with $\omega_{r,\overline{D}}(f,t) \leq \text{const.} \cdot \varphi(t)$.

Let $1 \leq j \leq N$ be fixed and $r \in \mathbb{N}$. Let $f \in AC(\overline{D})$ have $r-1$ existing derivatives at the vertices a_k, $k = 1, \ldots, N$, of the polygon. Consider for $k \neq j+1$ the polynomial $P_{j,k}$ of degree at most r, that interpolates f at the vertices a_j and a_k and $f', \ldots, f^{(r-1)}$ at the vertex a_k. For $k = j+1$ let $P_{j,j+1}$ denote the polynomial of degree at most $2r-1$ that interpolated $f, f', \ldots, f^{(r-1)}$ at both points a_j and a_{j+1}. We define

$$\begin{aligned} \delta_r(f,h) \;&:=\; \max_j \sum_{\substack{k=1 \\ k\neq j}}^{N} \left\{ \int_0^h \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right)\right|}{u}\, du \right. \\ &+\; \left. h^r \cdot \int_h^{2\pi} \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right)\right|}{u^{r+1}}\, du \right\}. \end{aligned}$$

Let $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index. Consider the corresponding quasipolynomial

$$\mathcal{P}_{\mathbf{n}}(f)(z) := \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}. \tag{6}$$

The coefficients $x_{n_j,r,m}$ are determined through the relations

$$x_{n_j,r,m} = \sum_{p=0}^{n_j} (-1)^p \binom{r}{p} J_{n_j,r,mp},$$

where $J_{n_j,r,k}$ are the Fourier coefficients of the generalized Jackson kernel

$$K_{n,r}(t) := \lambda_{n,r} \left(\frac{\sin Mt/2}{t/2} \right)^{2r} = \frac{J_{n,r,0}}{2} + \sum_{k=1}^{n} J_{n,r,k} \cos kt.$$

Here $n \in \mathbb{N}$, $r \geq 2$, $M := \lfloor \frac{n}{r} \rfloor$, and $\lambda_{n,r}$ is chosen such that

$$\frac{1}{2\pi} \int_0^{2\pi} K_{n,r}(t)\, dt = 1.$$

For the quasipolynomials (6) the following direct approximation theorem is true:

Theorem 2.1 *Let $f \in AH_r^{\omega_r}(\overline{D})$, where ω_r is a normal majorant with exponent $r \in \mathbb{N}$ satisfying the Stechkin condition*

$$\int_0^h \frac{\omega_r(f,t)}{t}\, dt + h^r \cdot \int_h^{2\pi} \frac{\omega_r(f,t)}{t^{r+1}}\, dt \quad \leq \quad c \cdot \omega_r(f,h) \tag{7}$$

for all $0 < h < \frac{2\pi}{r}$ and a positive constant c. Let f be $r-1$–times continuously differentiable at the vertices a_k, $k = 1, \ldots, N$, and

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0, \quad 0 \leq s \leq r-1.$$

Let $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index.

Then we have for approximation with the quasipolynomial $\mathcal{P}_{\mathbf{n}}(f)$ weighted with the generalized Jackson kernel

$$\|f - \mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})} \leq \text{const.} \sum_{k=1}^{N} \Omega_r \left(\frac{1}{n_k} \right),$$

where Ω_r — a normal majorant with exponent r — satisfies inequality

$$\Omega_r(h) \leq \text{const.} \cdot \{\omega_r(h) + \delta_r(f,h)\}. \tag{8}$$

The proof is given in [1].

In the following section, we give Yu. I. Mel'nik's approach to the question, if this rate of approximation can be preserved, when we substitute the integral in (3) or in (4) by an appropriate sum.

3 Substitution of integrals by appropriate sums

In [4], Yu. I. Mel'nik proposed to substitute the Leont'ev coefficients in (2) by

$$\kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) = \frac{1}{\lambda_m^{(j)}} \sum_{k=1}^{N} d_k \frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} f\left(a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}\right) \left(e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p+1}{\widehat{n}}} - e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p}{\widehat{n}}}\right) \quad (9)$$

for all $\widehat{n} \in \mathbb{N}$. He considered functions $f \in AH_1^{\omega}(\overline{D})$ with $\sum_{n=1}^{N} d_k f(a_k) = 0$ and approximated them with partial series of the form

$$S_{n,\widehat{n}}(f)(z) = \sum_{m=1}^{n_0} \kappa_f(\lambda_m)\,\frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n} \kappa_f^{(\widehat{n})}(\lambda_m^{(j)})\,\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}.$$

For the rate of approximation Mel'nik reached (see [4])

$$\|f - S_{n,\widehat{n}}(f)\|_{AC(\overline{D})} \le \text{const.} \cdot \left\{\omega\left(\frac{1}{\widehat{n}}\right) + \omega\left(\frac{1}{n}\right)\right\} \ln n. \quad (10)$$

The question that remains open, is, how (9) can be extended, such that an estimate for the rate of approximation can be reached for arbitrary moduli?

If we have a closer look at (9) and compare this formula with (3) and (4), we see that the integral there is decomposed in $\widehat{n}$ integrals of length $\frac{2\pi}{\widehat{n}}$:

$$\begin{aligned} &\int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda_m^{(j)} \frac{a_j-a_k}{2\pi}\,\theta}\, d\theta \quad = \\ &= \frac{1}{\widehat{n}} \int_p^{\widehat{n}-1} \int_{2\pi\frac{p}{\widehat{n}}}^{2\pi\frac{p+1}{\widehat{n}}} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda_m^{(j)} \frac{a_j-a_k}{2\pi}\,\theta}\, d\theta. \end{aligned} \quad (11)$$

The exponential function can be integrated easily. In general, this is not the case for f: The antiderivative might not be known explicitly, or highly oscillating f may cause numerical problems. Therefore the term $f(a_k + \frac{a_j-a_k}{2\pi}\,\theta)$ is estimated by the value at the lower bound of the integral $f(a_k + (a_j - a_k)\frac{p}{\widehat{n}})$:

$$\kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) = \sum_{k=1}^{N} d_k \frac{a_k - a_j}{2\pi} \frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} f\left(a_k + (a_j - a_k)\frac{p}{\widehat{n}}\right) \int_{2\pi\frac{p}{\widehat{n}}}^{2\pi\frac{p+1}{\widehat{n}}} e^{-\lambda_m^{(j)} \frac{a_j-a_k}{2\pi}\theta}\, d\theta. \quad (12)$$

Evaluating the integral explicitly gives (9).

To get a better rate of approximation with coefficients of this special form we have to find a better approximation of the function f on the staight-line interval $[\,a_j, a_k\,]$. We give a solution to this problem in the following section.

4 Higher order approximation

In this section we consider the question, if a better choice of $\kappa_f^{(\widehat{n})}(\lambda_m^{(j)})$ allows a higher rate of approximation and estimation with r-th moduli of smoothness, $r \in \mathbb{N}$. The key to this problem is the estimation of f in (11). We substitute f by the value of the modified r-th difference operator

$$\Delta^r_{\frac{2\pi}{r\widehat{n}}} f(z) - f\left(z + \frac{2\pi}{\widehat{n}}\right) = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} f\left(z + k\,\frac{2\pi}{r\widehat{n}}\right). \tag{13}$$

For $r = 1$ this expression yields $\Delta^1_{\frac{2\pi}{r\widehat{n}}} f(z) - f(z + \frac{2\pi}{\widehat{n}}) = f(z)$. If we put $z = a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}$ here, we get Mel'nik's formulas (9) and (12).

Substituting f in (11) with (13) for $z = a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}$ and arbitrary $r \in \mathbb{N}$ yields

$$\begin{aligned} \kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) \;&=\; \frac{1}{\lambda_m^{(j)}} \sum_{k=1}^{N} d_k \,\frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} f\left(a_k + \frac{a_j - a_k}{\widehat{n}}\,\left(p + \frac{k}{r}\right)\right) \\ &\cdot\; \left(e^{-\lambda_m^{(j)}(a_j - a_k)\frac{p+1}{\widehat{n}}} - e^{-\lambda_m^{(j)}(a_j - a_k)\frac{p}{\widehat{n}}}\right). \end{aligned} \tag{14}$$

Now we can formulate the following approximation theorem:

Theorem 4.1 *Let ω_r be a normal majorant with exponent r satisfying the Stechkin condition (7). Let $f \in AH_r^{\omega_r}(\overline{D})$ and*

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0 \quad \textit{for all} \;\; 0 \le s \le r-1.$$

Let $\mathbf{n} = (n_1, \ldots, n_N)$ and $\widehat{\mathbf{n}} = (\widehat{n}_1, \ldots, \widehat{n}_N)$, $\mathbf{n}, \widehat{\mathbf{n}} \in \mathbb{N}^N$, be multi-indices.
Consider the partial series $\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}$ weighted with the generalized Jackson kernel

$$\begin{aligned} \mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z) \;&=\; \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} \\ &+\; \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, \kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}, \end{aligned} \tag{15}$$

where $\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})$ as in (14).
Then the approximation of f with $\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)$ yields

$$\|f - \mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)\|_{AC(\overline{D})} \le \text{const.} \cdot \left\{ \sum_{k=1}^{N} \Omega_r\left(\frac{1}{n_k}\right) + \sum_{k=1}^{N} \omega_r\left(\frac{1}{\widehat{n}_k}\right) \ln n_k \right\},$$

where Ω_r as in (8).

PROOF. We show that

$$\|\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)-\mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})}\le C\sum_{k=1}^{N}\omega_r\left(\frac{1}{\widehat{n}_k}\right)\ln n_k.$$

with $C>0$ independent of f, $\mathbf{n}$ and $\widehat{\mathbf{n}}$ and conclude with Theorem 2.1.

It is by (6) and (15) for all $z\in\overline{D}$

$$\begin{aligned}\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z)-\mathcal{P}_{\mathbf{n}}(f)(z) &= \\ &= \sum_{j=1}^{N}\sum_{m=n(j)}^{n_j}(1-x_{n_j,r,m})\left(\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})-\kappa_f(\lambda_m^{(j)})\right)\frac{e^{\lambda_m^{(j)}z}}{L'(\lambda_m^{(j)})}. \qquad (16)\end{aligned}$$

We have a closer look at the difference $\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})-\kappa_f(\lambda_m^{(j)})$. Using (4), (12) and (14) we get

$$\begin{aligned}\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})-\kappa_f(\lambda_m^{(j)}) &= \\ &= \frac{1}{\lambda_m^{(j)}}\sum_{k=1}^{N}d_k\frac{1}{\widehat{n}_j}\sum_{p=0}^{\widehat{n}_j-1}\sum_{l=0}^{r-1}(-1)^l\binom{r}{l}f\left(a_k+\frac{a_j-a_k}{\widehat{n}_j}\left(p+\frac{l}{r}\right)\right)\left(e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p+1}{\widehat{n}_j}}\right. \\ &\quad \left.-\,e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p}{\widehat{n}_j}}\right)-\frac{1}{2\pi}\sum_{k=1}^{N}d_k\,(a_k-a_j)\int_0^{2\pi}f\left(a_k+\frac{a_j-a_k}{2\pi}\,\theta\right)e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\,d\theta \\ &= \sum_{k=1}^{N}d_k\left\{\frac{a_k-a_j}{2\pi}\frac{1}{\widehat{n}_j}\sum_{p=0}^{\widehat{n}_j-1}\sum_{p=0}^{r-1}(-1)^l\binom{r}{l}f\left(a_k+\frac{a_j-a_k}{\widehat{n}_j}\left(p+\frac{l}{r}\right)\right)\right. \\ &\quad \cdot\int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}}e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\,d\theta \\ &\quad \left.-\,\frac{1}{2\pi}\,(a_k-a_j)\sum_{p=0}^{\widehat{n}_j-1}\int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}}f\left(a_k+\frac{a_j-a_k}{2\pi}\,\theta\right)e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\,d\theta\right\} \\ &= \sum_{k=1}^{N}d_k\,\frac{a_k-_j}{2\pi}\sum_{p=0}^{\widehat{n}_j-1}\int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}}\left\{\sum_{l=0}^{r-1}(-1)^l\binom{r}{l}f\left(a_k+\frac{a_j-a_k}{\widehat{n}_j}\left(p+\frac{l}{r}\right)\right)\right. \\ &\quad \left.-\,f\left(a_k+\frac{a_j-a_k}{2\pi}\,\theta\right)\right\}e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\,d\theta.\end{aligned}$$

Thus for the series (16) we can write

$$\begin{aligned}\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z)-\mathcal{P}_{\mathbf{n}}(f)(z) &= \\ &= \sum_{j=1}^{N}\sum_{m=n(j)}^{n_j}(1-x_{n_j,r,m})\sum_{k=1}^{N}d_k\frac{a_k-a_j}{2\pi}\sum_{p=0}^{\widehat{n}_j-1}\int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}}\left\{\sum_{l=0}^{r-1}(-1)^l\binom{r}{l}\right.\end{aligned}$$

$$\cdot\, f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right) - f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right)\Bigg\}\, e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\, d\theta\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N} d_k \frac{a_k - a_j}{2\pi} \sum_{p=0}^{\widehat{n}_j - 1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \Bigg\{ \sum_{l=0}^{r-1} (-1)^l \binom{l}{r} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right)$$

$$- f\left(a_k + \frac{a_k - a_j}{2\pi}\,\theta\right)\Bigg\} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\, d\theta.$$

Hence

$$\begin{aligned}
&\|\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f) - \mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})} \quad = \\
&\leq \max_{z\in\overline{D}} \sum_{j=1}^{N}\sum_{k=1}^{N} |d_k| \frac{|a_j - a_k|}{2\pi} \sum_{p=0}^{\widehat{n}_j - 1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \left| \sum_{l=0}^{r-1} (-1)^l \binom{r}{l} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right)\right.\\
&\quad \left. - f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right)\right| \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\right| d\theta\\
&\leq \max_{z\in\overline{D}} \sum_{j=1}^{N}\sum_{k=1}^{N} |d_k| \frac{|a_k - a_j|}{2\pi} \cdot \omega_r\left(f, \frac{|a_j - a_k|}{\widehat{n}_j}\right)\\
&\quad \cdot \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta} \cdot \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\right| d\theta \qquad (17)
\end{aligned}$$

Now it is enough to estimate the integral in (17). First let $k \neq j+1$. Then

$$\Re\left(i\,\frac{a_j - a_k}{a_{j+1} - a_j}\right) > 0. \qquad (18)$$

By (1) we infer

$$\left|e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\right| = \left|e^{-mi\frac{a_j-a_k}{a_{j+1}-a_j}\,\theta}\, e^{-q_j e^{i\beta_j}\frac{a_j-a_k}{2\pi}\,\theta}\, e^{-\delta_n^{(j)}\theta}\right| \leq C\, e^{-am\theta}$$

for some positive constants C, a and all $\theta \in [\,0, 2\pi\,]$. Thus we obtain for all $z \in \overline{D}$

$$\begin{aligned}
&\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\,\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\right| d\theta\\
&\leq \text{ const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-am\theta}\right| d\theta\\
&= \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \cdot \frac{e^{-2\pi am} - 1}{-am} \quad \leq \quad \text{const.} \ln(n_j), \qquad (19)
\end{aligned}$$

since the family $\left\{\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\right\}_{m\geq n(j)}$ and the generalized Jackson coefficients $1 - x_{n_j,r,m}$ are bounded.

For $k = j + 1$ we have for the integral in (17) and property b)

$$\begin{aligned}
&\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta} \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right| d\theta \\
\leq\ &\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta} \cdot \left| e^{-am} + A \cdot e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j + a_{j+1}}{2}\right)} \right| \right| d\theta \\
\leq\ &\left\{ \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta}\, e^{-am} \right| \right. \\
&\left. + \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta} \cdot \left| e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j + a_{j+1}}{2}\right)} \right| \right| d\theta \right\}. \qquad (20)
\end{aligned}$$

for constants $a, A > 0$. Hence for the first sum by (1)

$$\begin{aligned}
&\left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta}\, e^{-am} \right| \\
\leq\ &\text{const.} \cdot \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-am} e^{im\theta} \right| \quad \leq \quad C
\end{aligned}$$

independently of n_j. Now to the second sum in (20) by property a):

$$\begin{aligned}
&\left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi}\theta} \left| e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j + a_{j+1}}{2}\right)} \right| \right| \\
\leq\ &\text{const.} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{im\theta} \left| e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j + a_{j+1}}{2}\right)} \right| \right|. \qquad (21)
\end{aligned}$$

Further we can write with property a)

$$e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j + a_{j+1}}{2}\right)} = e^{\frac{2\pi m i}{a_{j+1} - a_j}\left(z - \frac{a_j + a_{j+1}}{2}\right)}\, e^{q_j e^{i\beta_j}\left(z - \frac{a_j + a_{j+1}}{2}\right)}. \qquad (22)$$

The second complex exponential on the right hand side in (22) can be estimated by the constant $\max_{z\in\overline{D}} e^{q_j e^{i\beta_j}\left(z - \frac{a_j - a_{j+1}}{2}\right)}$, which is independent of m. Because of (18) we have to

estimate the first exponential for $z \in [\, a_j, a_{j+1} \,]$ only. Let $z = a_j + \frac{a_{j+1}-a_j}{2\pi}(t+i\varepsilon)$, where $t \in [\, 0, 2\pi \,]$ and $\varepsilon \in]\, 0, \varepsilon_1 \,]$, $\varepsilon_1 > 0$. Then

$$e^{\frac{2\pi mi}{a_{j+1}-a_j}\left(a_j + \frac{a_{j+1}-a_j}{2\pi}(t+i\varepsilon) - \frac{a_j+a_{j+1}}{2}\right)} = e^{mi(t+i\varepsilon)}\, e^{2\pi mi}.$$

Thus

$$\begin{aligned}
& \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{mi\theta} \left| e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_j+a_{j+1}}{2}\right)} \right| \right| d\theta \\
\leq\ & \text{const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{mi\theta} \left| e^{mi(t+i\varepsilon)} \right| \right| d\theta \\
\leq\ & \text{const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\varepsilon m + im\theta} \right| d\theta \\
\leq\ & c\, \ln(n_j)
\end{aligned}$$

in D, where $c > 0$ can be chosen independently of ε. For $\varepsilon \to 0$ the claim follows for all $z \in \overline{D}$. □

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Convergence of a Finite Element Method for Scalar Conservation Laws with Boundary Conditions in Two Space Dimensions

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Abstract

In this paper, a finite element method for general scalar conservation laws is analyzed: convergence towards the unique solution is proved for two-dimensional space with initial and boundary conditions, by using a uniqueness theorem for measure valued solutions. The method has some advantages: it is an explicit finite element scheme, which is suitable for computing convection dominated flows and discontinuous solutions for multi-dimensional hyperbolic conservation laws. It is superior to other methods in some techniques which are flexible in dealing with convergence.

Keywords. finite element method, conservation law, convergence, measure-valued solution, uniqueness theorem, weighted energy estimate, superconvergence.

1 Introduction

In this paper convergence of a finite element method is proved for general scalar conservation laws in 2-D space with initial and boundary conditions. We use the concept of measure valued solutions to scalar conservation laws with initial and boundary conditions, uniqueness theorem about measure valued solutions proved in [8]. This uniqueness theorem, which is a generalization of the corresponding result for the pure initial value problem proved in [6] yields convergence in L_p norm, $1 \leq p \leq \infty$, towards the unique solution, for approximate solutions of a scalar conservation law provided they are: (A) uniformly bounded in the L^∞ norm; (B) weakly consistent with all entropy inequalities; (C) strongly consistent with the initial conditions. In section 4 the finite element method is proved to satisfy (A) and in section 6 the conditions (B) and (C) are verified. We note the convergence proof does not require estimates of the total variation, which is usually used together with classical compactness arguments to prove convergence of finite difference schemes. In [11], to guarantee maximum principle, it is required that the viscosity coefficient $\varepsilon = O(h)$. Comparing this scheme with the upwind finite element scheme in [11], the elements must be divided into two categories in this scheme: for the elements in the interior domain, the viscosity coefficient $\varepsilon_1 = O(h)$, but for the elements intersecting the boundary of the domain, the viscosity coefficient $\varepsilon_2 = O(h^{\frac{1}{2}})$ to guarantee the maximum principle and convergence.

An outline of the paper is as follows. In section 2, we quote the related result in [8]. In section 3, we introduce the finite element scheme. In section 4, we prove a maximum norm estimate. In

[10], the proof on the uniform boundedness of $\|u_h\|_{L^\infty}$ is rather technical. For our method, by the nature of the explicit scheme and its monotone property, we can greatly simplify its proof. In section 5, we prove energy estimate for the stability in L_2 norm. In section 6, we prove the convergence; Lemma 6.2 plays a critical role. Some valuable techniques are used in this paper, such as superconvergence estimate, weighted energy estimate and L_2 stability, which play a key role in the convergence analysis. Some good ideas from [1], [12], [7], [2], [4] and [3] are also helpful to construct the scheme. Numerical experiments, to be reported elsewhere, have shown that the scheme gives satisfactory results.

2 Measure valued solutions with boundary condition

In this section, we quote the definition of measure valued solutions of scalar conservation laws with initial and boundary conditions and the following uniqueness result for measure valued solutions in [8], which still hold under our assumptions. The proof of convergence of the finite element solutions will be based on Theorem 2.1 below.

Let Ω be a bounded open set of R^d with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. The outward unit normal n exists almost everywhere on Γ. The mathematical prolem is to find $u : \Omega \times R_+ \to R$ satisfying the conservation law

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = 0 \quad \text{in} \quad \Omega \times R_+, \tag{1}$$

the initial condition

$$u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega, \tag{2}$$

and the boundary condition: for all $k \in R$, $\quad a.e.\ (\overline{x}, t) \in \Gamma \times R_+$:

$$(\text{sgn}\ (u(\overline{x}, t) - k) - \text{sgn}\ (a(\overline{x}, t) - k))(f(u(\overline{x}, t)) - f(k)) \cdot n(\overline{x}) \geq 0. \tag{3}$$

Here $f = (f_1, f_2, ..., f_d)$: $R \to R^d$, $a : \Gamma \times R_+ \to R$, $u_0 : \Omega \to R$, are given smooth functions, and the function $\text{sgn} : R \to R$ is defined by

$$\text{sgn}\ (x) = \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Let $\{u_j\}$ be a uniformly bounded sequence in $L_\infty(\Omega \times R_+)$, i.e. for some constant K,

$$\|u_j\|_{L_\infty(\Omega \times R+)} \leq K, \quad j = 1, 2, 3, ... \tag{4}$$

Then according to Young's theorem there exists a subsequence, still denoted by $\{u_j\}$, and an associated measurable measure valued mapping $\nu_{(\cdot)} : \Omega \times R_+ \to Prob(R)$ such that

$$\text{supp}\ \nu_{(x,t)} \subset \{\lambda : |\lambda| \leq K\} \quad a.e.\ (x, t) \in \Omega \times R_+, \tag{5}$$

and $\forall g \in C(R)$, the $L_\infty(\Omega \times R_+)$ weak star limit $g(u_j(\cdot)) \overset{*}{\rightharpoonup} \overline{g}(\cdot)$, $j \to \infty$, exists, where

$$\overline{g}(x, t) = \int_R g(\lambda) d\nu_{(x,t)}(\lambda) \equiv \langle \nu_{(x,t)}, g(\lambda) \rangle \quad a.e.\ (x, t) \in \Omega \times R_+. \tag{6}$$

A Young measure ν, associated with a sequence $\{u_j\}$ satisfying (4), is called a *measure valued solution* (mv-solution) to (1)–(3) if for all $\phi \in C_0^1(\overline{\Omega} \times R_+)$, $\phi \geq 0$, and for all $k \in R$, we have

$$\int_{\Omega \times R_+} \langle \nu_{(x,t)}, |\lambda - k| \rangle \phi_t + \langle \nu_{(x,t)}, (\text{sgn } (\lambda - k))(f(\lambda) - f(k)) \rangle \cdot \nabla \phi dx dt$$
$$- \int_{\Gamma \times R_+} \langle \gamma \nu_{(\overline{x},t)}, f(\lambda) - f(k) \rangle \cdot n(\overline{x}) \phi \, \text{sgn}\,(a-k) ds dt \geq 0,$$
$$\lim_{t \to 0} \int_{\Omega} \langle \nu_{(x,t)}, |\lambda - u_0| \rangle dx = 0.$$

We introduce the following uniqueness result for mv-solutions and trace theorem proved in [8].

Theorem 2.1. *Suppose that a Young measure ν associated with the sequence $\{u_j\}$ is a mv-solution to (1)–(3) and let w denote the unique BV-solution of (1)–(3). Then $\nu_{(x,t)} = \delta_{w(x,t)}$ a.e., i.e., $\nu_{(x,t)}$ reduces a.e. to the Dirac measure concentrated at $w(x,t)$, and the sequence $\{u_j\}$ converges strongly in $L_1^{loc}(\Omega \times R_+)$ to w.*

Lemma 2.2. *(Trace Theorem) Let $\nu : \Omega \times R_+ \to Prob(R)$ be a Young measure associated with a sequence $\{u_j\}$ satisfying (4). Then there are a sequence $\{y_j \in (0, \varepsilon)\}$ with $y_j \to 0$ and a measurable Young measure $\gamma\nu : \Gamma \times R_+ \to Prob(R)$ such that supp $\gamma\nu_{(\overline{x},t)} \subset \{\lambda : |\lambda| \leq K\}$ a.e. $(\overline{x}, t) \in \Gamma \times R_+$, and for every $g \in C(R)$, the $L^\infty(\Gamma \times R_+)$ weak star limit $\langle \nu_{(x(\cdot,y_j),\cdot)}, g(\lambda) \rangle \overset{*}{\rightharpoonup} \overline{g}(\cdot,\cdot)$, as $j \to \infty$ exists, i.e.*

$$\lim_{j \to \infty} \int_{\Gamma \times R_+} \langle \nu_{(x(\overline{x},y_j),t)}, g(\lambda) \rangle \varphi ds dt = \int_{\Gamma \times R_+} \overline{g}(\overline{x}, t) \varphi ds dt \quad \forall \, \varphi \in L^1(\Gamma \times R_+), \tag{7}$$

where ds is the Lebesgue measure on Γ, and for a.e. $(\overline{x}, t) \in \Gamma \times R_+$,

$$\overline{g}(\overline{x}, t) = \int_R g(\lambda) d\gamma\nu_{(\overline{x},t)} \equiv \langle \gamma\nu_{(\overline{x},t)}, g(\lambda) \rangle. \tag{8}$$

3 Formulation of the finite element method

Let Ω be a polygonal domain in R^2 with a Lipschitz continuous boundary $\Gamma = \partial\Omega$. We assume that $\sup_{y \in R} |f''(y)| < \infty$. This is not a severe restriction since the exact solution is bounded and thus $f(y)$ may be modified for large $|y|$ if necessary. Below we denote by C a positive constant independent of h, not necessarily the same at each occurrence. Let $0 \leq t_0 < t_1 < ... < t_N = T^*$ be a sequence of time levels, $I_n = (t_n, t_{n+1})$, $S_n = \Omega \times I_n$ and $\Omega_n = \Omega \times \{t_n\}$, $\triangle t = t_{n+1} - t_n$. Let Ω_h be a quasi-uniform triangulation of Ω. Denote $\mathcal{S}_1 = \{T \mid T \cap \Gamma = \Phi\}$ and $\mathcal{S}_2 = \{T \mid T \cap \Gamma \neq \Phi\}$. Let h_T be the diameter of element T and $h = \max_T h_T$. The shape function is continuous and linear on each T. Let φ_i be the shape function associated with the node x_i, $\varphi_i(x_j) = \delta_{ij}$. For given node x_i, let $\mathcal{T}_i$ be the set of elements neighboring x_i, $\Omega_i = \cup_{T \in \mathcal{T}_i} T$, and I_i be the index set of the nodes of $T \subset \Omega_i$ besides x_i. Introduce the sets $\mathcal{T}_{1,i} = \{T \in \mathcal{T}_i \mid T \cap \Gamma = \Phi\}$ and $\mathcal{T}_{2,i} = \{T \in \mathcal{T}_i \mid T \cap \Gamma \neq \Phi\}$, where Φ is the empty set. Denote $u_i^n = u(x_i, n\Delta t)$. Let $\delta > 0$ be a sufficiently small constant, $h \leq \delta$, and

$$\triangle t = O(h^{\frac{3}{2}}), \quad \varepsilon_1 = h\delta^{-\frac{1}{4}}, \quad \varepsilon_2 = h^{\frac{1}{2}}\delta^{\frac{1}{4}}. \tag{9}$$

ε_1 and ε_2 will be used as artificial viscosity constants.

We then define some quantities

$$K_{1i}^n = -\frac{\Delta t}{A_i} \sum_{T\in\mathcal{T}_{1,i}} \sum_{j\in I_i} a_{ij}^T J_1(u_i^n, u_j^n) h_{ij},$$
$$K_{2i}^n = -\frac{\Delta t}{A_i} \sum_{T\in\mathcal{T}_{2,i}} \sum_{j\in I_i} a_{ij}^T J_2(u_i^n, u_j^n) h_{ij},$$

where

$$a_{ij}^T = \int_T \nabla\varphi_i \cdot \nabla\varphi_j dx, \qquad A_i = \int_{\Omega_i} \varphi_i dx,$$
$$\psi_1 = -\int_{x_i}^x \frac{f(u)\cdot\tau_E}{\varepsilon_1 u}\, ds \quad \text{if} \quad E\cap\Gamma = \Phi, \qquad \psi_2 = -\int_{x_i}^x \frac{f(u)\cdot\tau_E}{\varepsilon_2 u}\, ds \quad \text{if} \quad E\cap\Gamma \neq \Phi,$$
$$J_1(u_i^n, u_j^n) = \frac{(e^{\psi_1}u)_j^n - (e^{\psi_1}u)_i^n}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1}\, ds}, \qquad J_2(u_i^n, u_j^n) = \frac{(e^{\psi_2}u)_j^n - (e^{\psi_2}u)_i^n}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2}\, ds}.$$

We note $a_{ij}^T \le 0$ for $i \neq j$, $|a_{ij}^T| \le C$. The point $x \in E$, E is the edge of T connecting x_i and x_j, $\int_{x_i}^x \cdot ds$ denotes a line integral from x_i to x, τ_E is the unit vector pointing from x_i to x_j, and h_{ij} is the length of E. For $i = 1, 2$, $(e^{\psi_i}u)_j^n$ is the value of $e^{\psi_i}u$ at the point of $(x_j, n\Delta t)$. The scheme is as follows:

$$u_i^{n+1} = u_i^n + K_{1i}^n + K_{2i}^n, \quad n \ge 0 \tag{10}$$
$$u_i^0 = \int_{\Omega_i} u_0\varphi_i\, dx/A_i. \tag{11}$$
$$u_i^n = a_i^n, \qquad \text{if} \quad x_i \in \Gamma, n > 0. \tag{12}$$

Without loss of generality we let $\overline{a} : \overline{\Omega} \times [0, T^*] \to R$ be a smooth extension of a. $u_0 \in L^\infty(\Omega), \text{supp} u_0 \subset\subset \Omega$. Interpolating $\overline{a}$ linearly on each element T, we get $\overline{a}_h^n (n \ge 0)$. Then we extend $\overline{a}_h^n (n \ge 0)$ to the whole domain $\overline{\Omega} \times [0, T^*] = \{(x, t) \mid x \in \overline{\Omega}, t \in [0, T^*]\}$ such that it keeps constant on $[n\Delta t, (n+1)\Delta t)$, $\forall n$, which is denoted by $\overline{a}_h$. Similarly, by using the value u_i^n on each node, and interpolating linearly on each element T, we extend the solution to (10) to the whole domain $\overline{\Omega} \times [0, T^*]$ such that it is constant on $[n\Delta t, (n+1)\Delta t)$, $\forall n$, denoted by u_h. Let

$$v_h = u_h - \overline{a}_h, \quad V_h^n = \{v \mid v \in H^1(S_n), v|_T \in P_1(T), v|_{\Gamma\times R_+} = 0\}. \tag{13}$$

Then $v_h \in V_h = \Pi_{n\ge 0} V_h^n$. Let $\pi : \Pi_{n\ge 0} C(S_n) \to V_h$ be the usual linear interpolation operator. The main result is the following.

Theorem 3.1. *We assume that $m \le u_0 \le M$, $m \le \overline{a} \le M$. The functions u_h converge strongly in $L_1^{loc}(\Omega \times R_+)$ to the unique BV-solution of (1)–(3) as $h \to 0$.*

We introduce some lemmas to prove Theorem 3.1. The lemmas are easy to prove, and we only show Lemma 3.5. For notational convenience, we omit the superscript n.

Lemma 3.2. *If u is a constant function, namely, $u_i = u_j = u$, then*

$$J_1(u, u) = J_2(u, u) = -f(u)\cdot\tau_E. \tag{14}$$

Lemma 3.3. *If $u, v \in P_1(T)$, and c is a constant vector, then*

$$\int_T \nabla u \cdot \nabla v \, dx = \sum_{i<j} a_{ij}^T (u_i - u_j)(v_j - v_i), \tag{15}$$

$$\int_T c \cdot \nabla v \, dx = \sum_{i<j} a_{ij}^T c \cdot \tau_E h_{ij} (v_i - v_j), \tag{16}$$

where u_i, u_j, v_i, v_j are the values of u, v at nodes x_i and x_j.

Lemma 3.4. *Assume that u_i, $u_j \in [m, M]$ and (9) holds. Then*

$$J_1(u_i, u_j) = \frac{e^{\psi_{1j}}(u_j - u_i)}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} - f(u_i) \cdot \tau_E + u_i O(u_j - u_i), \tag{17}$$

$$J_2(u_i, u_j) = \frac{e^{\psi_{2j}}(u_j - u_i)}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2} ds} - f(u_i) \cdot \tau_E + u_i O(u_j - u_i), \tag{18}$$

where $\psi_{1j} = \psi_1(x_j)$, $\psi_{2j} = \psi_2(x_j)$.

Lemma 3.5. *Assume that u_i, $u_j \in [m, M]$ and (9) holds. Then*

$$\frac{\partial J_1}{\partial u_j} = \frac{\varepsilon_1 e^{\psi_{1j}}}{\int_{x_i}^{x_j} e^{\psi_1} ds} \{1 + O(\frac{h_{ij}}{\varepsilon_1})\}, \quad \frac{\partial J_1}{\partial u_i} = O(\frac{\varepsilon_1}{h_{ij}}), \tag{19}$$

$$\frac{\partial J_2}{\partial u_j} = \frac{\varepsilon_2 e^{\psi_{2j}}}{\int_{x_i}^{x_j} e^{\psi_2} ds} \{1 + O(\frac{h_{ij}}{\varepsilon_2})\}, \quad \frac{\partial J_2}{\partial u_i} = O(\frac{\varepsilon_2}{h_{ij}}). \tag{20}$$

Proof: By the definition of J_1 and J_2, we have

$$\begin{aligned}
\frac{\partial J_1}{\partial u_i} = & \frac{e^{\psi_{1j}} \int_{x_i}^{x_j} \left(-\frac{f(u)}{\varepsilon_1 u}\right)' \frac{|x_j - x|}{h_{ij}} \cdot \tau_E \, ds u_j - 1}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} \\
& - \frac{(e^{\psi_{1j}} u_j - u_i) \int_{x_i}^{x_j} e^{\psi_1} \left(\int_{x_i}^{x} \left(-\frac{f(u)}{\varepsilon_1 u}\right)' \frac{|x_j - x|}{h_{ij}} \cdot \tau_E \, ds \right) ds}{\left(\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds \right)^2}, \\
\frac{\partial J_1}{\partial u_j} = & \frac{\varepsilon_1 e^{\psi_{1j}}}{\int_{x_i}^{x_j} e^{\psi_1} ds} \Bigg\{ 1 + \int_{x_i}^{x_j} \left(-\frac{f(u)}{\varepsilon_1 u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds u_j \\
& - \frac{(u_j - e^{-\psi_{1j}} u_i) \int_{x_i}^{x_j} e^{\psi_1} \left(\int_{x_i}^{x} \left(-\frac{f(u)}{\varepsilon_1 u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds \right) ds}{\int_{x_i}^{x_j} e^{\psi_1} ds} \Bigg\}.
\end{aligned}$$

Since u_i, $u_j \in [m, M]$ and δ is sufficiently small,

$$\frac{\partial J_1}{\partial u_i} = O(\frac{\varepsilon_1}{h_{ij}}), \quad \frac{\partial J_1}{\partial u_j} = \frac{\varepsilon_1 e^{\psi_{1j}}}{\int_{x_i}^{x_j} e^{\psi_1} ds} \{1 + O(\frac{h_{ij}}{\varepsilon_1})\}.$$

The relation (20) is proved similarly. □

Lemma 3.6. *If $v \in C^1(\overline{\Omega})$, then $\lim_{\eta \to 0} \int_{\{x \in \Omega \,|\, |v(x)| < \eta\}} |\nabla v| dx = 0$.*

4 Maximum norm estimate

Lemma 4.1. *Under the assumptions of Lemma 3.5, if $u_i^n \in [m, M]$ and δ is sufficiently small, then $u_i^{n+1} \in [m, M]$.*

Proof: If $x_i \in \Gamma$, then $u_i^{n+1} = a_i^{n+1} \in [m, M]$. Now assume $x_i \notin \Gamma$. Applying Lemmas 3.5, 3.2, 3.3, and the condition $u_i^n \in [m, M]$, for sufficiently small δ, we have

$$\begin{aligned}
\frac{\partial u_i^{n+1}}{\partial u_i} &\geq 1 - C\frac{\Delta t}{A_i}(\sup_j \sum_{T\in\mathcal{T}_{1,i}} a_{ij}^T)\frac{\varepsilon_1}{h}h - C\frac{\Delta t}{A_i}(\sup_j \sum_{T\in\mathcal{T}_{2,i}} a_{ij}^T)\frac{\varepsilon_2}{h}h = 1 + O(\delta^{\frac{1}{4}}) \geq 0,\\
\frac{\partial u_i^{n+1}}{\partial u_j} &\geq -\frac{\Delta t\varepsilon_1}{A_i}(\sum_{T\in\mathcal{T}_{1,i}} a_{ij}^T)\frac{e^{\psi_{1j}}h_{ij}}{\int_{x_i}^{x_j} e^{\psi_1}\,ds}\left\{1 + O(\frac{h_{ij}}{\varepsilon_1})\right\} - \frac{\Delta t\varepsilon_2}{A_i}(\sum_{T\in\mathcal{T}_{2,i}} a_{ij}^T)\frac{e^{\psi_{2j}}h_{ij}}{\int_{x_i}^{x_j} e^{\psi_2}\,ds}\left\{1 + O(\frac{h_{ij}}{\varepsilon_2})\right\}\\
&= C(\delta^{\frac{1}{4}} + \frac{h^{\frac{1}{2}}}{\delta^{\frac{1}{4}}})(1 + O(\delta^{\frac{1}{4}})) \geq 0,\\
u_i^{n+1} &\geq m + \frac{\Delta t}{A_i}\sum_{T\in\mathcal{T}_{1,i}}\sum_{j\in I_i} a_{ij}^T f(m)\tau_E h_{ij} + \frac{\Delta t}{A_i}\sum_{T\in\mathcal{T}_{2,i}}\sum_{j\in I_i} a_{ij}^T f(m)\tau_E h_{ij}\\
&= m + \frac{\Delta t}{A_i}\sum_{T\in\mathcal{T}_i}\int_T f(m)\nabla\varphi_i dx\\
&= m + \frac{\Delta t}{A_i}\int_{\Omega_i} f(m)\nabla\varphi_i dx = m.
\end{aligned}$$

Similarly, $u_i^{n+1} \leq M$. □

5 Energy estimate

Lemma 5.1. *Under the assumption of Lemma 3.5, if δ is sufficiently small, then*

$$\begin{aligned}
&\frac{1}{2}\sum_i (v_i^{N+1})^2 A_i - \frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij}\\
&- \frac{\Delta t}{2}\sum_{n=0}^{N}\sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}\,ds}h_{ij}\\
&\leq C\{\frac{1}{2}\Delta t\sum_{n=0}^{N}\sum_i (\frac{\overline{a}_i^{n+1} - \overline{a}_i^n}{\Delta t})^2 A_i + \Delta t\sum_{n=0}^{N}\int_{\partial\Omega_h} |\overline{a}_h^n f(\overline{a}_h^n) - F(\overline{a}_h^n)|ds\\
&\quad + \frac{1}{2}\sum_i (v_i^0)^2 A_i + C_2\Delta t\varepsilon_2\sum_{n=0}^{N}\sum_{T\in\mathcal{S}_2}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx\\
&\quad + C_1\Delta t\varepsilon_1\sum_{n=0}^{N}\sum_{T\in\mathcal{S}_1}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx + C_3\Delta t\sum_{n=0}^{N}\sum_T\int_T |\nabla(\pi\overline{a}^{n+1})|dx\},
\end{aligned}$$

where we define $F(u)$ as the entropy flux with respect to $U(u) = u^2/2$, and by (13), $v_i^n = u_i^n - \overline{a}_i^n$.

Proof: We multiply (10) by $v_i^{n+1}A_i$, and sum over i. Write

$$\sum_i (K_{1i}^n + K_{2i}^n)(u_i^{n+1} - \overline{a}_i^{n+1})A_i = \sum_i \{K_{1i}^n(u_i^n - \overline{a}_i^{n+1}) + K_{2i}^n(u_i^n - \overline{a}_i^{n+1}) + (K_{1i}^n + K_{2i}^n)^2\}A_i.$$

Applying Lemma 3.4, we have

$$\begin{aligned}
&\sum_i K_{1i}(u_i^n - \overline{a}_i^{n+1})A_i \\
&= -\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \Big(\frac{e^{\psi_{1j}}(u_j^n - u_i^n)}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds} - f(u_i^n)\cdot\tau_E + u_i^n O(u_j^n - u_i^n)\Big)\cdot(u_i^n - u_j^n)h_{ij} \\
&\quad + \Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \Big(\frac{e^{\psi_{1j}}(u_j^n - u_i^n)}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds} - f(u_i^n)\cdot\tau_E + u_i^n O(u_j^n - u_i^n)\Big)(\overline{a}_i^{n+1} - \overline{a}_j^{n+1}))h_{ij} \\
&= \Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij} + \Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)(\overline{a}_i^{n+1} - \overline{a}_j^{n+1})}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij} \\
&\quad - \Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(-f(u_i^n)\tau_E + u_i O(u_j^n - u_i^n))(u_i^n - u_j^n)h_{ij} \\
&\quad + \Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(-f(u_i^n)\tau_E + u_i^n O(u_j^n - u_i^n))(\overline{a}_i^{n+1} - \overline{a}_j^{n+1})h_{ij}.
\end{aligned}$$

Changing the index from 1 to 2, we get a similar expression for $\sum_i K_{2i}^n(u_i^n - a_i^{n+1})A_i$. Here and below we let u_T^n be the mean value of u^n on T. By Lemma 3.2 and the definition of $F(U)$, we have

$$\begin{aligned}
\int_{\Omega_h}\nabla\cdot F(u^n)dx &= \int_{\partial\Omega_h} n\cdot F(\overline{a}_h^n)ds = \int_{\Omega_h} u^n\nabla\cdot f(u^n)dx \\
&= -\int_{\Omega_h} f(u^n)\cdot\nabla u^n dx + \int_{\partial\Omega_h}\overline{a}_h^n f(\overline{a}_h^n)\cdot n ds \\
&= \int_{\partial\Omega_h}\overline{a}_h^n f(\overline{a}_h^n)\cdot n ds - \sum_{T\in\mathcal{S}_1}\int_T f(u^n)\cdot\nabla u^n dx \\
&\quad - \sum_{T\in\mathcal{S}_2}\int_T f(u^n)\cdot\nabla u^n dx \\
&= \int_{\partial\Omega_h}\overline{a}_h^n f(\overline{a}_h^n)\cdot n ds - \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T f(u_T^n)\cdot\tau_E h_{ij}(u_i^n - u_j^n) \\
&\quad - \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T f(u_T^n)\cdot\tau_E h_{ij}(u_i^n - u_j^n).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_i K_{1i}^n(u_i^n-\overline{a}_i^{n+1})A_i+\sum_i K_{2i}^n(u_i^n-\overline{a}_i^{n+1})A_i\\
&=\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}+\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{2j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}ds}h_{ij}\\
&\quad+\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}\\
&\quad+\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{2j}}(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}ds}h_{ij}\\
&\quad-\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T(f(u_T^n)-f(u_i^n))\tau_E+u_i^nO(u_j^n-u_i^n))h_{ij}(u_i^n-u_j^n)\\
&\quad-\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j}a_{ij}^T(f(u_T^n)-f(u_i^n))\tau_E+u_i^nO(u_j^n-u_i^n))h_{ij}(u_i^n-u_j^n)\\
&\quad+\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T(f(u_T^n)-f(u_i^n))\tau_E+u_i^nO(u_j^n-u_i^n))h_{ij}(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})\\
&\quad+\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j}a_{ij}^T(f(u_T^n)-f(u_i^n))\tau_E+u_i^nO(u_j^n-u_i^n))h_{ij}(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})\\
&\quad+\Delta t\int_{\partial\Omega_h}(\overline{a}_h^nf(\overline{a}_h^n)-F(\overline{a}_h^n))\cdot nds-\Delta t\sum_T a_{ij}^Tf(u_T^n)\cdot\tau_Eh_{ij}(\overline{a}_i^{n+1}-\overline{a}_j^{n+1}).
\end{aligned}$$

We will bound each of the terms. By Schwarz inequality, for δ small enough, we have

$$\begin{aligned}
&-\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}\\
&\le-\frac{1}{16}\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}-4\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(\overline{a}_j^{n+1}-\overline{a}_i^{n+1})^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}\\
&\le-\frac{1}{16}\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}ds}h_{ij}+C\varepsilon_1\Delta t\sum_{T\in\mathcal{S}_1}\int_T|\nabla\pi(\overline{a}_h^{n+1})|^2dx.
\end{aligned}$$

Changing the index from 1 to 2, we have a similar estimate for the term

$$-\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{2j}}(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}ds}h_{ij}.$$

Next,

$$\begin{aligned}
|f(u_T^n)-f(u_i^n)|&\le C|u_T^n-u_i^n|\le Ch_T|\nabla u^n|_{0,\infty,T}\\
&\le C|u^n|_{1,T}=C(-\sum_{l<m}a_{lm}^T(u_l^n-u_m^n)^2)^{\frac{1}{2}};\\
|u_j^n-u_i^n|&\le|u_i^n-u_T^n|+|u_j^n-u_T^n|\le C(-\sum_{l<m}a_{lm}^T(u_l^n-u_m^n)^2)^{\frac{1}{2}}.
\end{aligned}$$

Also,

$$
\begin{aligned}
&-\Delta t \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T(f(u_T^n)-f(u_i^n))\cdot\tau_E + u_i^n O(u_j^n-u_i^n))(u_i^n-u_j^n)h_{ij}\\
&\le -C\Delta t h \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T(u_i^n-u_j^n)^2 \le -C\Delta t\delta^{\frac{1}{4}}\sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T\frac{e^{\psi_{2j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}\,ds}h_{ij}\\
&\le -\frac{1}{16}\Delta t \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T\frac{e^{\psi_{2j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}\,ds}h_{ij},
\end{aligned}
$$

and a similar bound for the term

$$
-\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(f(u_T^n)-f(u_i^n))\cdot\tau_E + u_i^n O(u_j^n-u_i^n))(u_i^n-u_j^n)h_{ij}.
$$

Note that

$$
\begin{aligned}
&\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T u_i^n O(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})h_{ij}\\
&\le -\frac{1}{8}\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij} - 2\Delta t\varepsilon_1\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})^2\\
&= -\frac{1}{8}\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij} + 2\Delta t\varepsilon_1\sum_{T\in\mathcal{S}_1}\int_T|\nabla(\pi\overline{a}^{n+1})|^2dx.
\end{aligned}
$$

Changing the index from 1 to 2, we get an estimate for

$$
\Delta t \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T u_i^n O(u_j^n-u_i^n)(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})h_{ij}.
$$

By Lemma 4.1 and Lemma 3.3, we have

$$
-\Delta t\sum_T a_{ij}^T f(u_T^n)\cdot\tau_E h_{ij}(\overline{a}_i^{n+1}-\overline{a}_j^{n+1}) \le C_3\Delta t\sum_T\int_T|\nabla(\pi\overline{a}^{n+1})|dx.
$$

Furthermore,

$$
\begin{aligned}
&\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(f(u_T^n)-f(u_i^n))\cdot\tau_E(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})h_{ij}\\
&\le 2\Delta t\varepsilon_1\sum_{T\in\mathcal{S}_1}\int_T|\nabla(\pi\overline{a}^{n+1})|^2dx - \frac{1}{8}\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T\frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij},
\end{aligned}
$$

and we have a similar estimate for the term

$$
\Delta t \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T(f(u_n^T)-f(u_i^n))\cdot\tau_E(\overline{a}_i^{n+1}-\overline{a}_j^{n+1})h_{ij}.
$$

Finally, we estimate

$$
\sum_i (K_{1i}^n+K_{2i}^n)^2A_i.
$$

Let $p(x) = f(u_i^n)(x - x_i)$, then $\triangle p = 0$. By Lemma 3.3, we have

$$\begin{aligned} 0 &= \int_{\Omega_i} \nabla p \cdot \nabla \varphi_i \, dx \\ &= \sum_{T \in \mathcal{T}_{1,i}} \sum_{j \in I_i} a_{ij}^T f(u_i^n) \cdot \tau_E h_{ij} + \sum_{T \in \mathcal{T}_{2,i}} \sum_{j \in I_i} a_{ij}^T f(u_i^n) \cdot \tau_E h_{ij}. \end{aligned}$$

Next, by Lemma 3.4 we have

$$\begin{aligned} &|K_{1i}^n + K_{2i}^n| \\ &= |\frac{\Delta t}{A_i} \{ \sum_{T \in \mathcal{T}_{1,i}} \sum_{j \in I_i} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} \, ds} h_{ij} \\ &\quad + \sum_{T \in \mathcal{T}_{2,i}} \sum_{j \in I_i} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n - u_i^n)}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2} \, ds} h_{ij} + u_i^n O(u_j^n - u_i^n) h_{ij} \}| \\ &\leq C \frac{\Delta t}{A_i} \{ \sum_{T \in \mathcal{T}_{1,i}} \sum_{j \in I_i} (h + \varepsilon_1) |a_{ij}^T| |u_j^n - u_i^n| + \sum_{T \in \mathcal{T}_{2,i}} \sum_{j \in I_i} (h + \varepsilon_2) |a_{ij}^T| |u_j^n - u_i^n| \}, \end{aligned}$$

which implies that

$$\begin{aligned} &\sum_i |K_{1i}^n + K_{2i}^n|^2 A_i \\ &\leq 2C \sum_i \frac{\Delta t^2}{A_i} \{ \sum_{T \in \mathcal{T}_{1,i}} \sum_{j \in I_i} (h + \varepsilon_1) |a_{ij}^T| |u_j^n - u_i^n| \}^2 \\ &\quad + 2C \sum_i \frac{\Delta t^2}{A_i} \{ \sum_{T \in \mathcal{T}_{2,i}} \sum_{j \in I_i} (h + \varepsilon_2) |a_{ij}^T| |u_j^n - u_i^n| \}^2 \\ &\leq 2C\varepsilon_1 \Delta t (\delta^{\frac{1}{4}} h^{\frac{1}{2}} + 2h^{\frac{1}{2}} + h^{\frac{1}{4}}) \sum_{T \in \mathcal{S}_1} \sum_{i<j} |a_{ij}^T| (u_j^n - u_i^n)^2 \\ &\quad + 2C\varepsilon_2 \Delta t (h^{\frac{3}{4}} + 2h^{\frac{1}{2}} + \delta^{\frac{1}{4}}) \sum_{T \in \mathcal{S}_2} \sum_{i<j} |a_{ij}^T| (u_j^n - u_i^n)^2 \\ &\leq -\frac{1}{8} \Delta t \sum_{T \in \mathcal{S}_1} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} \, ds} h_{ij} - \frac{1}{8} \Delta t \sum_{T \in \mathcal{S}_2} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2} \, ds} h_{ij}. \end{aligned}$$

Combining all estimates above, we get

$$\begin{aligned} &\sum_i (K_{1i}^n + K_{2i}^n)(u_i^{n+1} - \overline{a}_i^{n+1}) A_i \\ &\leq \frac{1}{2} \Delta t \sum_{T \in \mathcal{S}_1} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} \, ds} h_{ij} + \Delta t \int_{\partial \Omega_h} |\overline{a}_h^n f(\overline{a}_h^n) - F(\overline{a}_h^n)| ds. \\ &\quad + \frac{1}{2} \Delta t \sum_{T \in \mathcal{S}_2} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2} \, ds} h_{ij} + C_1 \Delta t \varepsilon_1 \sum_{T \in \mathcal{S}_1} \int_T |\nabla(\pi \overline{a}^{n+1})|^2 dx \\ &\quad + C_2 \Delta t \varepsilon_2 \sum_{T \in \mathcal{S}_2} \int_T |\nabla(\pi \overline{a}^{n+1})|^2 dx + C_3 \Delta t \sum_T \int_T |\nabla(\pi \overline{a}^{n+1})| dx. \end{aligned}$$

On the other hand,

$$\sum_i (u_i^{n+1}-u_i^n)v_i^{n+1} = \sum_i \{(v_i^{n+1})^2 - v_i^n v_i^{n+1} + v_i^{n+1}(\overline{a}_i^{n+1}-\overline{a}_i^n)\},$$

$$\frac{1}{\Delta t}\sum_i \Delta t v_i^{n+1}(\overline{a}_i^{n+1}-a_i^n) \geq -\frac{1}{2}\Delta t\{\sum_i (v_i^{n+1})^2 + \sum_i (\frac{\overline{a}_i^{n+1}-\overline{a}_i^n}{\Delta t})^2\},$$

$$-\sum_i v_i^n v_i^{n+1} \geq -\frac{1}{2}(\sum_i (v_i^{n+1})^2 + \sum_i (v_i^n)^2),$$

$$\frac{1}{2}\sum_i \{(v_i^{n+1})^2 - (v_i^n)^2\} - \frac{1}{2}\Delta t\{\sum_i \{(v_i^{n+1})^2 + (\frac{\overline{a}_i^{n+1}-\overline{a}_i^n}{\Delta t})^2\}\}$$
$$\leq \sum_i (u_i^{n+1}-u_i^n)v_i^{n+1}.$$

Hence

$$\frac{1}{2}\sum_i (v_i^{n+1})^2 A_i - \frac{1}{2}\Delta t \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds} h_{ij}$$
$$-\frac{1}{2}\Delta t \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}\,ds} h_{ij}$$
$$\leq \frac{1}{2}\Delta t\{\sum_i (v_i^{n+1})^2 A_i + \sum_i (\frac{\overline{a}_i^{n+1}-\overline{a}_i^n}{\Delta t})^2 A_i\} + \frac{1}{2}\sum_i (v_i^n)^2 A_i$$
$$+ C_1\Delta t\varepsilon_1 \sum_{T\in\mathcal{S}_1}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx + C_2\Delta t\varepsilon_2 \sum_{T\in\mathcal{S}_2}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx$$
$$+ C_3\Delta t\sum_T \int_T |\nabla(\pi\overline{a}^{n+1})|dx + \Delta t\int_{\partial\Omega_h} |\overline{a}_h^n f(\overline{a}_h^n) - F(\overline{a}_h^n)|ds.$$

By summing them up with respect to n, $n = 0, 1, ...N$, we get

$$\frac{1}{2}\sum_i (v_i^{N+1})^2 A_i - \frac{1}{2}\Delta t\sum_{n=0}^N \sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds} h_{ij}$$
$$-\frac{1}{2}\Delta t\sum_{n=0}^N \sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(u_j^n-u_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}\,ds} h_{ij}$$
$$\leq \frac{1}{2}\Delta t\sum_{n=0}^N \{\sum_i (v_i^{n+1})^2 A_i + \sum_i (\frac{\overline{a}_i^{n+1}-\overline{a}_i^n}{\Delta t})^2 A_i\} + \frac{1}{2}\sum_i (v_i^0)^2 A_i$$
$$+ C_3\Delta t\sum_{n=0}^N \sum_T \int_T |\nabla(\pi\overline{a}^{n+1})|dx + C_2\Delta t\varepsilon_2 \sum_{n=0}^N \sum_{T\in\mathcal{S}_2}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx$$
$$+ C_1\Delta t\varepsilon_1 \sum_{n=0}^N \sum_{T\in\mathcal{S}_1}\int_T |\nabla(\pi\overline{a}^{n+1})|^2 dx + \Delta t\sum_{n=0}^N \int_{\partial\Omega_h} |\overline{a}_h^n f(\overline{a}_h^n) - F(\overline{a}_h^n)|ds.$$

Now applying a discrete Gronwall inequality yields the conclusion of Lemma 5.1. □

6 Proof of convergence

To prove Theorem 3.1, we first note that by Lemma 4.1, the discrete solutions u_h are uniformly bounded in L^∞ norm. Thus, the sequence $\{u_h\}$ satisfies (4). Then there exists according to Young's theorem a Young measure $\nu_{(\cdot)} : \Omega \times R_+ \to Prob(R)$ associated with a subsequence $\{u_{h_j}\}$ such that ν satisfies (5)–(6). By Lemma 2.2, there exists an associated Young measure $\gamma\nu_{(\cdot)} : \Gamma \times R_+ \to Prob(R)$ satisfying (8). Introduce the following notation:

$$\overset{\circ}{S}_n = \{(x,t) \mid x \in \mathcal{S}_1, t \in I_n\}, \quad \partial S_n = S_n \backslash \overset{\circ}{S}_n,$$
$$\overset{\circ}{S}^N = \bigcup_{n=0}^{N} \overset{\circ}{S}_n, \qquad \partial S^N = \bigcup_{n=0}^{N} \partial S_n.$$

Let $\mathrm{sg}_\eta = \mathrm{sgn} * \omega_\eta \in C^\infty(R)$ be the standard mollification of sgn, where

$$\omega_\eta(y) = \frac{\omega(\frac{y}{\eta})}{\eta}, \quad \omega \in C_0^\infty(-1,1), \quad \omega \geq 0, \quad \int_R \omega dy = 1, \quad \mathrm{sg}'_\eta(s) = 2\omega_\eta(s) \geq 0.$$

Further, $\phi \in C_0^\infty(\overline{\Omega} \times (0,T^*))$, $\phi \geq 0$, $\chi \in V_h$ with χ linear on T, is constant on $[n\Delta t, (n+1)\Delta t), \forall n$, and $\chi|_{\overset{\circ}{S}^N} = 1$. Proofs of Lemmas 6.1 and 6.2 will be given later.

Lemma 6.1. *The Young measure ν associated with $\{u_{h_j}\}$ is a mv-solution in the interior domain:*

$$\frac{\partial}{\partial t}\langle \nu_{(x,t)}, |\lambda - k| \rangle + \frac{\partial}{\partial t}\langle \nu_{(x,t)}, (sgn\ (\lambda - k))(f(\lambda) - f(k)) \rangle \leq 0 \quad \forall\, k \in R, \text{ in } D'(\Omega \times R_+).$$

Lemma 6.2. *The Young measure $\gamma\nu$ associated with ν given in Lemma* 6.1 *satisfies*

$$\langle \gamma\nu_{(\overline{x},t)}, (sgn\ (\lambda - k) - sgn\ (a - k))(f(\lambda) - f(k)) \rangle \cdot n \geq 0 \quad \forall\, k \in R, \text{ in } D'(\Gamma \times R_+).$$

Lemma 6.3. *(Superconvergence) If $w \in W^{1,\infty}(\Omega)$, then for $r = 0, 1$,*

$$\|w - \pi w\|_{W^{r,\infty}(\Omega)} \leq Ch^{1-r}\|w\|_{W^{1,\infty}(\Omega)},$$
$$\|vw - \pi(vw)\|_{W^{r,\infty}(\Omega)} \leq Ch^{1-r}\|v\|_{L^\infty(\Omega)}\|w\|_{W^{1,\infty}(\Omega)} \quad \forall\, v \in V_h.$$

Proof: It is sufficient to consider one triangle $T \in \Omega$. We use standard interpolation error estimate in [5]:

$$\|w - \pi w\|_{0,\infty,T} \leq Ch|w|_{1,\infty,T} \leq Ch\|w\|_{1,\infty,T},$$
$$\|w - \pi w\|_{1,\infty,T} \leq Ch^0|w|_{1,\infty,T} \leq C\|w\|_{1,\infty,T}.$$

For $\phi|_T \in L^1(T)$, define $\mathcal{P}\phi = \int_T \phi dx / \int_T dx$, the L^2 projection of ϕ. Then

$$\|w - \mathcal{P}w\|_{0,\infty,T} \leq Ch\|\nabla w\|_{L^\infty}(T),$$
$$\|vw - \pi(vw)\|_{0,\infty,T} \leq \|v\mathcal{P}w - \pi(v\mathcal{P}w)\|_{0,\infty,T} + \|(I - \pi)(v(w - \mathcal{P}w))\|_{0,\infty,T}.$$

For $v \in V_h$, $\mathcal{P}w|_T$ is constant, which implies $\pi(v\mathcal{P}w) = \mathcal{P}w\pi v = v\mathcal{P}w$. By the inverse inequality and above interpolation error estimate, we have

$$\begin{aligned}
\|(I-\pi)(v(w-\mathcal{P}w))\|_{0,\infty,T} &\le Ch\|v(w-\mathcal{P}w)\|_{1,\infty,T} \\
&\le C\|v(w-\mathcal{P}w)\|_{0,\infty,T} \le Ch\|\nabla(v(w-\mathcal{P}w))\|_{0,\infty,T} \\
&= Ch\|(w-\mathcal{P}w)\nabla v + v\nabla(w-\mathcal{P}w)\|_{0,\infty,T} \\
&\le Ch(\|\nabla v\|_{0,\infty,T}\|w-\mathcal{P}w\|_{0,\infty,T} + \|w-\mathcal{P}w\|_{1,\infty,T}\|v\|_{0,\infty,T}) \\
&\le Ch(\|v\|_{0,\infty,T}\|w\|_{1,\infty,T} + h\|\nabla w\|_{0,\infty,T}\|v\|_{1,\infty,T}) \\
&\le Ch(\|v\|_{0,\infty,T}\|w\|_{1,\infty,T} + h\|w\|_{1,\infty,T}\frac{\|v\|_{0,\infty,T}}{h}) \\
&= Ch\|v\|_{0,\infty,T}\|w\|_{1,\infty,T}.
\end{aligned}$$

Thus $\|vw - \pi(vw)\|_{0,\infty,T} \le Ch\|v\|_{0,\infty,T}\|w\|_{1,\infty,T}$. By the inverse inequality, we have

$$\|vw-\pi(vw)\|_{1,\infty,T} \le \frac{C}{h}\|vw-\pi(vw)\|_{0,\infty,T} \le C\|v\|_{0,\infty,T}\|w\|_{1,\infty,T}.$$

□

Now we begin with convergence analysis. Taking

$$U(u_i^{n+1}) = (u_i^{n+1}-k)\mathrm{sg}_\eta(u_i^{n+1}-k), \quad \psi_i^n = \mathrm{sg}_\eta(u_i^n-k)\phi_i^n \cdot \chi_i^n,$$

multiplying (10) by ψ_i^n and summing up, we get

$$\begin{aligned}
&\sum_n\sum_i \frac{(u_i^{n+1}-k)-(u_i^n-k)}{\Delta t}\psi_i^n \Delta t A_i + \sum_n\sum_i\sum_{T\in\mathcal{T}_{1,i}}\sum_{j\in I_i}\Delta t a_{ij}^T J_1(u_i^n,u_j^n)h_{ij}\psi_i^n \\
&+\sum_n\sum_i\sum_{T\in\mathcal{T}_{2,i}}\sum_{j\in I_i}\Delta t a_{ij}^T J_2(u_i^n,u_j^n)h_{ij}\psi_i^n = 0.
\end{aligned}$$

By (4)–(6) and Lemma 4.1 and Lemma 3.6, as $\eta \to 0$ and $h \to 0$,

$$\begin{aligned}
&\sum_n\sum_i \frac{(u_i^{n+1}-k)-(u_i^n-k)}{\Delta t}\psi_i^n\Delta t A_i \\
&= \sum_n\sum_i U(u_i^{n+1})\frac{\phi_i^n\chi_i^n - \phi_i^{n+1}\chi_i^{n+1}}{\Delta t}A_i \\
&\quad + \sum_i\sum_n \frac{(u_i^{n+1}-k)\mathrm{sg}_\eta(u_i^n-k) - (u_i^{n+1}-k)\mathrm{sg}_\eta(u_i^{n+1}-k)}{\Delta t}\phi_i^n\chi_i^n\Delta t A_i \\
&\to -\int_{\Omega\times R_+}\langle \nu_{(x,t)}, (\lambda-k)\mathrm{sgn}(\lambda-k)\rangle\cdot\phi_t dxdt.
\end{aligned}$$

Consider

$$\begin{aligned}
&\sum_n\sum_i\frac{\Delta t}{A_i}A_i\sum_{T\in\mathcal{T}_{1,i}}\sum_{j\in I_i}a_{ij}^T J_1(u_i,u_j)h_{ij}\psi_i^n + \sum_n\sum_i\frac{\Delta t}{A_i}A_i\sum_{T\in\mathcal{T}_{2,i}}\sum_{j\in I_i}a_{ij}^T J_2(u_i,u_j)h_{ij}\psi_i^n \\
&\quad \equiv A+B+C+D+E,
\end{aligned}$$

where

$$A = \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T (J_1(u_i,u_j) - J_1(u_i,u_i)) h_{ij} \mathrm{sg}_\eta(u_i^n - k)((\phi\chi)_i^n - (\phi\chi)_j^n),$$

$$B = \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T (J_1(u_i,u_j) - J_1(u_i,u_i)) h_{ij} \mathrm{sg}_\eta(u_i^n - k)((\phi\chi)_i^n - (\phi\chi)_j^n),$$

$$C = \Delta t \sum_n \sum_{T} \sum_{i<j} (-a_{ij}^T) f(u_i^n) \cdot \tau_E h_{ij} \mathrm{sg}_\eta(u_i^n - k)((\phi\chi)_i^n - (\phi\chi)_j^n),$$

$$D = \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T J_1(u_i^n,u_j^n) h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n,$$

$$E = \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T J_2(u_i^n,u_j^n) h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n.$$

We consider each of these five terms.

$$A = \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T \frac{\partial J_1}{\partial u_j}(u_j^n - u_i^n) h_{ij} \mathrm{sg}_\eta(u_i^n - k)(\phi_i^n - \phi_j^n)\chi_i^n,$$

$$\begin{aligned} B &= \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T \frac{\partial J_2}{\partial u_j}(u_j^n - u_i^n) h_{ij} \mathrm{sg}_\eta(u_i^n - k)(\phi_i^n - \phi_j^n)\chi_j^n \\ &\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T \frac{\partial J_2}{\partial u_j}(u_j^n - u_i^n) h_{ij} \mathrm{sg}_\eta(u_i^n - k)\phi_i^n(\chi_i^n - \chi_j^n) \\ &\equiv B_I + B_{II}, \end{aligned}$$

where $\frac{\partial J_1}{\partial u_j}, \frac{\partial J_2}{\partial u_j}$ are mean values determined by intermediate value theorem. By Cauchy's inequality, interpolation error estimate Lemma 5.1, Lemma 3.5, and Lemma 4.1, we get

$$\begin{aligned} |A| &\le C\{\Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} (-a_{ij}^T) \frac{e^{\psi_{1j}}(u_j^n - u_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} h_{ij}\}^{\frac{1}{2}} \cdot \\ &\qquad \{\Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} \varepsilon_1 (-a_{ij}^T) \frac{\int_{x_i}^{x_j} e^{\psi_1} ds}{e^{\psi_{1j}} h_{ij}} \cdot (\phi_j^n - \phi_i^n)^2 (\chi_i)^2\}^{\frac{1}{2}} \\ &\le C\{\Delta t \varepsilon_1 \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} (-a_{ij}^T)(\phi_i^n - \phi_j^n)^2\}^{\frac{1}{2}} \\ &= C\{\Delta t \varepsilon_1 \sum_n \int_{\Omega_h} |\nabla(\pi\phi^n)|^2 dx\}^{\frac{1}{2}} \\ &= C\{\Delta t \varepsilon_1 (\sum_n \int_{\Omega_h} |\nabla(\phi^n - \pi\phi^n)|^2 dx + \sum_n \int_{\Omega_h} |\nabla\phi^n|^2 dx)\}^{\frac{1}{2}} \\ &\le C\varepsilon_1^{\frac{1}{2}} (\|\phi\|_{1,\Omega\times[0,T^*]} + h\|\phi\|_{2,\Omega\times[0,T^*]}) \to 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

Similarly, $B_I \to 0$ as $h \to 0$. Let

$$\begin{aligned} C_{III} &= -\Delta t \sum_n \sum_T \int_T (f(u^n) - f(k)) \mathrm{sg}_\eta(u^n - k) \nabla(\pi(\phi^n \chi)) dx \\ &= -\Delta t \sum_n \sum_T \int_T f(u^n) \mathrm{sg}_\eta(u^n - k) \nabla(\pi(\phi^n \chi)) dx \\ &\quad + \Delta t \sum_n \sum_T \int_T f(k) \mathrm{sg}_\eta(u^n - k) \nabla(\pi(\phi^n \chi)) dx \\ &= -\Delta t \sum_n \sum_T \int_T f(u^n) \mathrm{sg}_\eta(u^n - k) \nabla(\pi(\phi^n \chi)) dx \\ &\quad - \Delta t \sum_n \sum_T \int_T f(k) \nabla(\mathrm{sg}_\eta(u^n - k)) \pi(\phi^n \chi) dx \\ &\equiv C_I + C_{II}. \end{aligned}$$

By Lemma 3.6, $C_{II} \to 0$, as $\eta \to 0$. On the other hand,

$$C_I = -\Delta t \sum_n \sum_T \sum_{i<j} a_{ij}^T f(u_T^n) \mathrm{sg}_\eta(u_T^n - k) \tau_E h_{ij} ((\phi\chi)_i^n - (\phi\chi)_j^n),$$

where u_T^n is a mean value on T. To guarantee convergence, we will prove $|C - C_I| \to 0$ as $\eta \to 0$, $h \to 0$ in Proposition 2 later. On the other hand, in order to prove Lemma 6.1, we analyze

$$\begin{aligned} C_{III} &= \Delta t \sum_n \sum_T \int_T (f(u^n) - f(k)) \mathrm{sg}_\eta(u^n - k) \nabla((I - \pi)(\phi^n \chi)) dx \\ &\quad - \Delta t \sum_n \sum_T \int_T (f(u^n) - f(k)) \mathrm{sg}_\eta(u^n - k) (\nabla \phi^n \chi + \nabla \chi \phi^n) dx \\ &\equiv C_{I'} + C_{II'}. \end{aligned}$$

$$\begin{aligned} C_{I'} &= \Delta t \sum_n \sum_{T \in \mathcal{S}_1} \int_T (f(u^n) - f(k)) \mathrm{sg}_\eta(u^n - k) \nabla((I - \pi)(\phi^n \chi)) dx \\ &\quad + \Delta t \sum_n \sum_{T \in \mathcal{S}_2} \int_T (f(u^n) - f(k)) \mathrm{sg}_\eta(u^n - k) \nabla((I - \pi)(\phi^n \chi)) dx \\ &\equiv C_a + C_b. \end{aligned}$$

Similar to the estimate in A, we have

$$|C_a| \le C \|(I - \pi)\phi^n\|_{1,\Omega_h} \le Ch|\phi|_{2,\Omega \times R_+} \to 0 \quad \text{as} \quad h \to 0.$$

$$
\begin{aligned}
D &= \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T J_1(u_i^n, u_j^n) h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T J_1(u_i^n, u_i^n) h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T J_1(u_i^n, u_i^n) h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \\
&= -\Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T (u_i^n - u_j^n)^2 \mathrm{sg}_\eta'(\overline{u} - k)(\phi\chi)_j^n h_{ij} \frac{\varepsilon_1 e^{\psi_{1j}}}{\int_{x_i}^{x_j} e^{\psi_1} ds} (1 + O(\frac{h}{\varepsilon_1})) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T f(u_i^n) \cdot \tau_E h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \\
&\equiv D_I + D_{II},
\end{aligned}
$$

$$
\begin{aligned}
E &= -\Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T (u_i^n - u_j^n)^2 \mathrm{sg}_\eta'(\overline{u} - k)(\phi\chi)_i^n h_{ij} \frac{\varepsilon_2 e^{\psi_{2j}}}{\int_{x_i}^{x_j} e^{\psi_2} ds} (1 + O(\frac{h}{\varepsilon_2})) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T f(u_i^n) \cdot \tau_E h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \\
&\equiv E_I + E_{II}.
\end{aligned}
$$

For δ is sufficiently small, $\mathrm{sg}_\eta'(\overline{u} - k) \geq 0, \quad h \leq \delta, \quad 1 + O(\frac{h}{\varepsilon_1}) \geq 0, \quad 1 + O(\frac{h}{\varepsilon_2}) \geq 0$. Thus $D_I \geq 0$, $E_I \geq 0$, $D_I + E_I \geq 0$. By Lemma 3.6, we prove easily as $\eta \to 0$,

$$
E_{II} + D_{II} = -\Delta t \sum_n \sum_T \sum_{i<j} a_{ij}^T f(u_i^n) \tau_E h_{ij} (\mathrm{sg}_\eta(u_i^n - k) - \mathrm{sg}_\eta(u_j^n - k))(\phi\chi)_j^n \to 0.
$$

With the above preparations, we now prove Lemma 6.1.

Proof of Lemma 6.1: Take $\phi \in C_0^\infty(\Omega \times R_+)$, $\phi \geq 0$. For h is sufficiently small, we have $B_{II} = 0$, $C_b = 0$, $\nabla\chi\phi^n = 0$. As $\eta \to 0$, $h \to 0$, by the dominated convergence theorem and (4)–(6), we get

$$
C_{II'} \to -\int_{\Omega\times R_+} \langle \nu_{(x,t)}, (f(\lambda) - f(k))\mathrm{sgn}(\lambda - k)\rangle \nabla\phi dx dt, \tag{21}
$$

$$
\int_{\Omega\times R_+} (\langle \nu_{(x,t)}, (\lambda - k)\mathrm{sgn}(\lambda - k)\rangle \phi_t + \langle \nu_{(x,t)}, (f(\lambda) - f(k))\mathrm{sgn}(\lambda - k)\rangle \nabla\phi) dx dt \geq 0. \tag{22}
$$

□

We shall use Lemma 6.4 (weighted energy estimate) below to estimate v_h near the boundary. We first discuss the continuous model of Lemma 6.4. Since Ω is polygonal, for a.e. $\overline{x} \in \Gamma$, there exists a positive ε_0, and the change of coordinate: $x = \overline{x} - yn(\overline{x}), \quad (\overline{x}, y) \in \Gamma \times (0, \varepsilon_0), \quad n(\overline{x})$ is the outward unit normal of Γ at $\overline{x}$. $\nabla y = -n(\overline{x})$, where $y = |x - \overline{x}| = \mathrm{dist}\,(x, \Gamma) = \hat{d}_x$, which is uniformly Lipschitz continuous. Let

$$
\overline{d}_x = \begin{cases} \hat{d}_x, & x \in \Omega \\ 0, & x \notin \Omega. \end{cases}
$$

Taking $\tilde{\varepsilon} < \frac{h}{40}$ and letting $d_x^{\tilde{\varepsilon}} = \overline{d}_x * \rho_{\tilde{\varepsilon}}$ be a standard mollification of $\overline{d}_x$, we have

$$\begin{aligned} |d_x^{\tilde{\varepsilon}} - \hat{d}_x| &= |\int_{|x-y|<\tilde{\varepsilon}} \rho_{\tilde{\varepsilon}}(x-y)(\hat{d}_x - \hat{d}_y)dy| \\ &\leq \int_{|x-y|<\tilde{\varepsilon}} \rho_{\tilde{\varepsilon}}(x-y)|x-y|dy \\ &\leq |x-y_1| \leq \tilde{\varepsilon}, \quad y_1 \in B(x,\tilde{\varepsilon}). \end{aligned}$$

The unique BV-solution of (1)–(3) is bounded. Let C_M be a constant such that $|v| \leq C_M$, $\forall (x,t) \in \Omega \times (0,T^*)$ and define β as follows:

$$\beta^{-1} = \sup_{|w|\leq C_M,} \frac{4|F(w,x,t)|}{w^2},$$

where $v = u - \overline{a}$, $f'(u) = (f_1'(u), f_2'(u))^T$, and

$$F = (F_1(v,x,t), F_2(v,x,t))^T = (\int_0^v f_1'(w+\overline{a})wdw, \int_0^v f_2'(w+\overline{a})wdw)^T.$$

Further we introduce the direction $\overline{\beta} = (1+\beta^2)^{-\frac{1}{2}}(\beta,1) = (\beta_1,\beta_2) \in R^2$, and for $n = 0,1,2,...$, we define the cut-off function $\psi : S_n \to R_+$,

$$\psi|_{S_n} = \begin{cases} e^{-\frac{\beta_1(d_x^{\tilde{\varepsilon}}-h)+\beta_2(t-t_{n+1})}{\tau}}, & \beta_1(d_x^{\tilde{\varepsilon}}-h)+\beta_2(t-t_{n+1}) > 0, \\ 1, & \beta_1(d_x^{\tilde{\varepsilon}}-h)+\beta_2(t-t_{n+1}) \leq 0, \end{cases}$$

where there exists a sufficiently large constant C', such that $\tau = C'h^\alpha$, $\alpha \in (\frac{1}{2}, \frac{3}{4})$. Note that ψ equals one on ∂S_n and decays exponentially in $\overset{\circ}{S}_n$. We analyze the continuous model of Lemma 6.4 (weighted energy estimate) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla \cdot f(u) = \varepsilon \cdot \triangle u & \text{in} \quad \Omega \times R_+, \\ u|_{t=0} = u_0 & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = a(x,t) & \text{on} \quad \partial\Omega \times R_+, \end{cases} \tag{23}$$

where

$$\varepsilon|_{S_n} = \begin{cases} \varepsilon_1, & (x,t) \in \overset{\circ}{S}_n, \\ \\ \varepsilon_2, & (x,t) \in \partial S_n. \end{cases}$$

Multiplying the equation by $v\psi$ and integrating by parts on S_n, we analyze items respectively:

$$\begin{aligned} \int_{S_n} \frac{\partial u}{\partial t} v\psi dxdt &= \int_{S_n} \frac{\partial(\frac{v^2}{2})}{\partial t}\psi dxdt + \int_{S_n} v\frac{\partial \overline{a}}{\partial t}\psi dxdt, \\ \int_{S_n} \frac{\partial(\frac{v^2}{2})}{\partial t}\psi dxdt &= \int_{\Omega_{n+1}} \frac{v^2}{2}\psi dx - \int_{\Omega_n} \frac{v^2}{2}\psi dx - \int_{S_n} \frac{v^2}{2}\psi_t dxdt, \\ \int_{S_n} v\frac{\partial \overline{a}}{\partial t}\psi dxdt &\leq \int_{S_n} \frac{v^2\psi}{2} dxdt + \frac{1}{2}\int_{S_n} |\frac{\partial \overline{a}}{\partial t}|^2 \psi dxdt. \end{aligned}$$

Moreover,

$$\begin{aligned}\int_{S_n} \nabla \cdot f(u) v\psi dxdt &= \int_{S_n} f'(v+\overline{a})\nabla(v+\overline{a})v\psi dxdt \\ &= \int_{S_n} f'(v+\overline{a})\nabla v v\psi dxdt \\ &\quad + \int_{S_n} f'(v+\overline{a})\nabla\overline{a} v\psi dxdt; \\ \int_{S_n} f'(v+\overline{a})\nabla\overline{a} v\psi dxdt &\le \int_{S_n} \frac{v^2\psi}{2} dxdt + C\int_{S_n} |\nabla\overline{a}|^2\psi dxdt.\end{aligned}$$

Using the property of F, we get

$$\begin{aligned}F_{1x_1}(v,x,t) &= f_1'(v+\overline{a})vv_{x_1} + \int_0^v (f_1'(w+\overline{a}))'_{x_1} w dw, \\ F_{2x_2}(v,x,t) &= f_2'(v+\overline{a})vv_{x_2} + \int_0^v (f_2'(w+\overline{a}))'_{x_2} w dw, \\ \nabla\cdot F &= f'(v+\overline{a})\nabla v v + \int_0^v \sum_{i=1}^2 (f_i'(w+\overline{a}))'_{x_i} w dw \equiv F_I + F_{II}.\end{aligned}$$

Since $\sup_{y\in R}|f''(y)| \le C$, $F_{II} \le C\frac{v^2}{2}$. And

$$\int_{S_n} \nabla\cdot F\psi dxdt = \int_{S_n} f'(v+\overline{a})\nabla v v\psi dxdt + \int_{S_n}\int_0^v \sum_{i=1}^2 (f_i'(w+\overline{a}))'_{x_i} w dw\psi dxdt.$$

Since

$$\begin{aligned}\int_{S_n} \nabla\cdot F\psi dxdt &= \int_{\Gamma\times(t_n,t_{n+1})} F\psi\cdot n ds dt - \int_{S_n} F\nabla\psi dxdt, \\ v|_{\partial\Omega} = 0, \quad F|_{\partial\Omega} &= (0,0),\end{aligned}$$

we have

$$\begin{aligned}\int_{S_n} f'(v+\overline{a})\nabla v v\psi dxdt &= -\int_{S_n} F\nabla\psi dxdt \\ &\quad - \int_{S_n}\int_0^v \sum_{i=1}^2 (f_i'(w+\overline{a}))'_{x_i} w dw\psi dxdt.\end{aligned}$$

Moreover,

$$|-\int_{S_n}\int_0^v \sum_{i=1}^2 (f_i'(w+\overline{a}))'_{x_i} w dw\psi dxdt| \le C\int_{S_n} \psi v^2 dxdt.$$

Next, we analyze

$$-\int_{S_n} \varepsilon\triangle u(u-\overline{a})\psi dxdt = \int_{S_n} \varepsilon\nabla u\cdot v\nabla\psi dxdt + \int_{S_n} \varepsilon\nabla u\nabla v\psi dxdt.$$

Here and below $c > 0$ is a constant to be chosen sufficiently small. For a.e. $\overline{x} \in \Gamma$, we have $\nabla d_x^{\tilde{\varepsilon}} = (\nabla \overline{d}_x)^{\tilde{\varepsilon}} = (\nabla \hat{d}_x)^{\tilde{\varepsilon}} = (-n(\overline{x}))^{\tilde{\varepsilon}}$, therefore $|\nabla d_x^{\tilde{\varepsilon}}| \leq 1$. On the other hand, we note that

$$\nabla\psi = -\frac{\beta_1}{\tau}\nabla d_x^{\tilde{\varepsilon}}\psi, \quad \psi_t = -\frac{\beta_2}{\tau}\psi, \quad h \leq \delta, \quad \varepsilon_1 \leq \varepsilon_2,$$

δ is sufficiently small and C' is a sufficiently large constant, we have

$$\begin{aligned}
\frac{\beta_1}{4c\tau}\varepsilon_1^2 &= \frac{\beta_1 h}{4cC'h^\alpha \delta^{\frac{1}{4}}}\varepsilon_1 \\
&= \frac{\beta_1}{4cC'}(\frac{h}{\delta})^{\frac{1}{4}} h^{\frac{3}{4}-\alpha}\varepsilon_1 \\
&\leq \frac{\beta_1}{4cC'}h^{\frac{3}{4}-\alpha}\varepsilon_1 \leq \frac{1}{8}\varepsilon_1, \\
\int_{S_n} \varepsilon\nabla u v\nabla\psi dxdt &\leq c\int_{S_n} v^2|\nabla\psi|dxdt + \int_{S_n}\frac{\beta_1}{4c\tau}\varepsilon_1^2|\nabla u|^2\psi dxdt \\
&\leq c\int_{S_n} v^2|\nabla\psi|dxdt + \frac{1}{8}\int_{S_n}\varepsilon_1|\nabla u|^2\psi dxdt \\
&\leq c\int_{S_n}\frac{v^2\psi}{\tau}dxdt + \frac{1}{4}\int_{S_n}\varepsilon(|\nabla v|^2 + |\nabla\overline{a}|^2)\psi dxdt, \\
\int_{S_n}\varepsilon\nabla u\nabla v\psi dxdt &= \int_{S_n}\varepsilon|\nabla v|^2\psi dxdt + \int_{S_n}\varepsilon\nabla\overline{a}\nabla v\psi dxdt, \\
\int_{S_n}\varepsilon\nabla\overline{a}\nabla v\psi dxdt &\leq \frac{1}{4}\int_{S_n}\varepsilon|\nabla v|^2\psi dxdt + \int_{S_n}\varepsilon|\nabla\overline{a}|^2\psi dxdt.
\end{aligned}$$

Combining all above estimates, we have

$$\begin{aligned}
&\int_{\Omega_{n+1}}\frac{v^2\psi}{2}dx + \frac{1}{2}\int_{S_n}\varepsilon|\nabla v|^2\psi dxdt - \int_{S_n}(\frac{v^2}{2}\psi_t + F\nabla\psi)dxdt \\
&\leq \int_{\Omega_n}\frac{v^2\psi}{2}dx + C\int_{S_n} v^2\psi dxdt + \frac{1}{2}\int_{S_n}|\frac{\partial\overline{a}}{\partial t}|^2\psi dxdt \\
&\quad + C\int_{S_n}|\nabla\overline{a}|^2\psi dxdt + c\int_{S_n}\frac{v^2\psi}{\tau}dxdt.
\end{aligned}$$

Since $\beta^{-1} \geq 4|F|/v^2$, we have

$$\begin{aligned}
1 + \frac{2F}{v^2}\beta\nabla d_x^{\tilde{\varepsilon}} &\geq 1 - \frac{2|F|}{v^2}\beta|\nabla d_x^{\tilde{\varepsilon}}| \geq 1 - \frac{2|F|}{v^2}\beta \geq \frac{1}{2}, \\
-\int_{S_n}(\frac{v^2}{2}\psi_t + F\nabla\psi)dxdt &= -\int_{S_n}\frac{v^2}{2}(1, \frac{F}{\frac{v^2}{2}})(\psi_t, \nabla\psi)dxdt \\
&= \frac{1}{\tau}\int_{S_n}\frac{v^2}{2}(1, \frac{2F}{v^2})(1+\beta^2)^{-\frac{1}{2}}(1, \beta\nabla d_x^{\tilde{\varepsilon}})\psi dxdt \\
&\geq c\int_{S_n}\frac{v^2\psi}{\tau}dxdt.
\end{aligned}$$

Then,

$$\int_{\Omega_{n+1}} \frac{v^2\psi}{2}dx + \frac{1}{2}\int_{S_n} \varepsilon|\nabla v|^2\psi dxdt \le \int_{\Omega_n} \frac{v^2\psi}{2}dx + C\int_{S_n} v^2\psi dxdt + \frac{1}{2}\int_{S_n} |\frac{\partial \overline{a}}{\partial t}|^2\psi dxdt + C\int_{S_n} |\nabla \overline{a}|^2\psi dxdt.$$

By Growall's inequality, we have

$$\begin{aligned}
&\int_{\Omega_{n+1}} \frac{v^2\psi}{2}dx + \frac{1}{2}\int_{S_n} \varepsilon|\nabla v|^2\psi dxdt \\
&\quad \le C\{\int_{\Omega_n} \frac{v^2\psi}{2}dx + \frac{1}{2}\int_{S_n} |\frac{\partial \overline{a}}{\partial t}|^2\psi dxdt + C\int_{S_n} |\nabla \overline{a}|^2\psi dxdt\} \\
&\quad \le C(\int_{\Omega_n} \psi dx + \int_{S_n} \psi dxdt) \le C(h+\tau).
\end{aligned}$$

We can then establish the following local stability result.

Lemma 6.4. *(Weighted Energy Estimate) Under the assumptions of the continuous model and (9), we have for h sufficiently small,*

$$\begin{aligned}
&\frac{1}{2}\sum_i (v_i^{n+1})^2\psi_i^{n+1}A_i - \frac{\Delta t}{2}\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(v_j^n - v_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1}ds} h_{ij}\psi_i^n \\
&- \frac{\Delta t}{2}\sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(v_j^n - v_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_2}}{\varepsilon_2}ds} h_{ij}\psi_i^n \le C(h+\tau).
\end{aligned}$$

On the basis of Lemma 6.4, we have the following estimate of v_h near the boundary.

Lemma 6.5. *Under the assumptions of Lemma 6.4, there is a constant C such that for h sufficiently small,*

$$\|v_h\|_{0,\infty,\partial S^N} \le C(\frac{h+\tau}{\varepsilon_2})^{\frac{1}{2}},$$

which in particular implies that

$$\lim_{h\to 0}\|v_h\|_{0,\infty,\partial S^N} = 0.$$

Proof: By the definition of v_h, Lemma 6.4, and the property of affine family of finite elements, $\exists T_0 \in \partial S_n$, such that

$$\begin{aligned}
\|v_h\|_{0,\infty,\partial S_n} &= \|v_h\|_{0,\infty,T_0} \le C\|\hat{v}_h\|_{0,\infty,\hat{T}_0} \le C\|\hat{v}_h\|_{1,\hat{T}_0} \\
&\le C\|B\||detB|^{-\frac{1}{2}}|v_h|_{1,T_0} = C|v_h|_{1,T_0} \\
&\le C|v_h|_{1,\partial S_n} = C\{-\Delta t\sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T(v_j^n - v_i^n)^2\}^{\frac{1}{2}} \\
&\le C\{\frac{-1}{\varepsilon_2}\frac{\Delta t}{2}\sum_{T\in\mathcal{S}_2}\sum_{i<j} a_{ij}^T \frac{e^{\psi_{2j}}(v_j^n - v_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_2}}{\varepsilon_2}ds} h_{ij}\psi_i^n\}^{\frac{1}{2}} \\
&\le C(\frac{h+\tau}{\varepsilon_2})^{\frac{1}{2}} \to 0 \quad \text{as} \quad h\to 0,
\end{aligned}$$

which prove the Lemma 6.5. □

Let $\phi = \overline{\phi}\overline{\chi}_\xi$, where $0 \le \overline{\phi} \in C_0^\infty(\overline{\Omega} \times R_+)$, and for $\xi > 0$, we set

$$H_\xi = 1 + \frac{1}{2}(\text{sgn } (\cdot, -\frac{3}{4}\xi) - \text{sgn } (\cdot, +\frac{3}{4}\xi)) * \omega_{\frac{1}{4}\xi}, \quad \overline{\chi}_\xi = H_\xi(\overline{a} - k).$$

Then $\overline{\chi}_\xi \in C^\infty(\overline{\Omega} \times R_+)$, $\overline{\chi}_\xi(x,t) \in [0,1]$, and

$$\overline{\chi}_\xi = \begin{cases} 0, & |\overline{a}(x,t) - k| < \frac{\xi}{2}, \\ 1, & |\overline{a}(x,t) - k| \ge \xi. \end{cases}$$

For h small enough, we obtain by Lemma 6.5, $\|v_h\|_{L^\infty(\partial S^N)} \le C(\frac{h+\tau}{\varepsilon_2})^{\frac{1}{2}} \le \frac{\xi}{3} - \eta$ and

$$\text{sg}_\eta(u_h - k)\overline{\phi}\overline{\chi}_\xi\chi|_{\partial S^N} = \text{sgn } (\overline{a} - k)\overline{\phi}\overline{\chi}_\xi\chi|_{\partial S^N}.$$

Consider the change of coordinates $x \to (\overline{x}, y)$ for x in a neighborhood of Γ:

$$x = \overline{x} - yn(\overline{x}), \ a.e. \ (\overline{x}, y) \in \Gamma \times (0, \rho).$$

Let

$$J = |\frac{\partial x}{\partial(\overline{x}, y)}|, \quad \Omega' \subset\subset \Omega, \quad \text{dist } (\Omega', \partial\Omega) = \rho.$$

We define open sets $\Omega_1 = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > 2\rho/3\}$, $\Omega_2 = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \rho/3\}$, and let ϕ_1 be the characteristic function of Ω_1.

$$\phi_1 = \begin{cases} 1, & x \in \Omega_1, \\ \\ 0, & x \notin \Omega_1. \end{cases}$$

Let $\rho_{1\xi} * \phi_1 = \chi_\rho$ be the standard mollification of ϕ_1. For a.e. $(\overline{x}, y) \in \Gamma \times (0, \rho)$, we have $\nabla y = -n(\overline{x})$ and

$$\chi_\rho = \begin{cases} 1, & y \ge \rho, \\ g(y), & y \in [\frac{\rho}{3}, \rho], \\ 0, & y \in [0, \frac{\rho}{3}], \end{cases} \quad \text{and} \quad \nabla\chi_\rho = \begin{cases} -g'(y)n(\overline{x}), & y \in [\frac{\rho}{3}, \rho], \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

For h is sufficiently small, $\chi_\rho|_{\mathcal{S}_2} = 0$ and we obtain

$$B_{II} = \Delta t \sum_n \sum_{T \in \mathcal{S}_2} \sum_{i<j} a_{ij}^T \frac{\partial J_2}{\partial u_j^n}(u_j^n - u_i^n)h_{ij}\text{sgn } (\overline{a}_i^n - k)\phi_i^n\{(\chi_i^n - \chi_{\rho_i}^n) - (\chi_j^n - \chi_{\rho_j}^n)\}. \tag{25}$$

With the choice of ϕ, χ_ρ, we have the next result.

Lemma 6.6.

$$B_{II} \to -\int_{\Gamma \times R_+} sgn(a-k)\langle\gamma\nu_{(\overline{x},t)}, f(a) - f(\lambda)\rangle \cdot n\phi dsdt \quad as \quad h \to 0, \quad \eta \to 0. \tag{26}$$

Proof: Multiplying the scheme (10) with $\psi_i^n = \text{sgn }(\overline{a}_i^n - k)\phi_i^n(\chi_i - \chi_{\rho_i})A_i$, and summing over i, we get

$$\sum_n \sum_i u_i^{n+1} \frac{\psi_i^n - \psi_i^{n+1}}{\Delta t} \Delta t A_i + \Delta t \sum_n \sum_{T \in \mathcal{S}_1} \sum_{i<j} a_{ij}^T J_1(u_i^n, u_j^n) h_{ij} (\psi_i^n - \psi_j^n)$$
$$+ \Delta t \sum_n \sum_{T \in \mathcal{S}_2} \sum_{i<j} a_{ij}^T J_2(u_i^n, u_j^n) h_{ij} (\psi_i^n - \psi_j^n) = 0.$$

We analyze three items above respectively. First we consider

$$\begin{aligned} G &= \Delta t \sum_n \sum_T \int_T f(u^n) \nabla(\pi(\text{sgn }(\overline{a} - k)\phi(\chi - \chi_\rho))) dx \\ &= \Delta t \sum_n \sum_T \int_T f(u_T^n) \nabla(\pi(\text{sgn }(\overline{a} - k)\phi(\chi - \chi_\rho))) dx \\ &= \Delta t \sum_n \sum_T f(u_T^n) \tau_E h_{ij} a_{ij}^T \{\text{sgn }(\overline{a}_i^n - k)\phi_i^n(\chi_i^n - \chi_{\rho_i}^n) - \text{sgn }(\overline{a}_j^n - k)\phi_j^n(\chi_j^n - \chi_{\rho_j}^n)\} \\ &\equiv P + P_I + P_{II} + P_{III} + P_{IV} + P_V + P_{VI}, \end{aligned}$$

where

$$
\begin{aligned}
P &= \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} (J_1(u_i^n,u_j^n) - J_1(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_i^n - \chi_{\rho_i}^n) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} (J_1(u_i^n,u_j^n) - J_1(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_j^n - \chi_{\rho_j}^n) \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} (J_2(u_i^n,u_j^n) - J_2(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}(\overline{a}_i^n - k)\phi_i^n(\chi_i^n - \chi_{\rho_i}^n) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} (J_2(u_i^n,u_j^n) - J_2(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}(\overline{a}_i^n - k)\phi_i^n(\chi_j^n - \chi_{\rho_j}^n), \\
P_I &= -\Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} J_1(u_i^n,u_j^n) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_i^n - \chi_{\rho_i}^n) \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} J_1(u_i^n,u_j^n) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_j^n - k)\phi_j^n(\chi_j^n - \chi_{\rho_j}^n) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} J_2(u_i^n,u_j^n) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_i^n - \chi_{\rho_i}^n) \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} J_2(u_i^n,u_j^n) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_j^n - k)\phi_j^n(\chi_j^n - \chi_{\rho_j}^n), \\
P_{II} &= \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} (J_1(u_i^n,u_j^n) - J_1(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)(\phi_i^n - \phi_j^n)(\chi_j^n - \chi_{\rho_j}^n) \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} (J_2(u_i^n,u_j^n) - J_2(u_i^n,u_i^n)) h_{ij} a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)(\phi_i^n - \phi_j^n)(\chi_j^n - \chi_{\rho_j}^n), \\
P_{III} &= \Delta t \sum_n \sum_T \sum_{i<j} (f(u_T^n) - f(u_i^n)) \tau_E a_{ij}^T h_{ij} \text{sgn}\ (\overline{a}_i^n - k)(\phi_i^n - \phi_j^n)(\chi_j^n - \chi_{\rho_j}^n), \\
P_{IV} &= \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} J_1(u_i^n,u_j^n) h_{ij} a_{ij}^T (\text{sgn}\ (\overline{a}_i^n - k) - \text{sgn}\ (\overline{a}_j^n - k))\phi_j^n(\chi_j^n - \chi_{\rho_j}^n) \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} J_1(u_i^n,u_i^n) h_{ij} a_{ij}^T (\text{sgn}\ (\overline{a}_i^n - k) - \text{sgn}\ (\overline{a}_j^n - k))\phi_j^n(\chi_j^n - \chi_{\rho_j}^n) \\
&\quad + \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} J_2(u_i^n,u_j^n) h_{ij} a_{ij}^T (\text{sgn}\ (\overline{a}_i^n - k) - \text{sgn}\ (\overline{a}_j^n - k))\phi_j^n(\chi_j^n - \chi_{\rho_j}^n), \\
&\quad - \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} J_2(u_i^n,u_i^n) h_{ij} a_{ij}^T (\text{sgn}\ (\overline{a}_i^n - k) - \text{sgn}\ (\overline{a}_j^n - k))\phi_j^n(\chi_j^n - \chi_{\rho_j}^n), \\
P_V &= \Delta t \sum_n \sum_T \sum_{i<j} (f(u_T^n) - f(u_i^n)) \tau_E a_{ij}^T h_{ij} (\text{sgn}\ (\overline{a}_i^n - k) - \text{sgn}\ (\overline{a}_j^n - k))\phi_j^n(\chi_j^n - \chi_{\rho_j}^n), \\
P_{VI} &= \Delta t \sum_n \sum_T \sum_{i<j} (f(u_T^n) - f(u_i^n)) \tau_E a_{ij}^T h_{ij} \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n\{(\chi_i^n - \chi_{\rho_i}^n) - (\chi_j^n - \chi_{\rho_j}^n)\}.
\end{aligned}
$$

It is trivial to prove that as $h \to 0$, $P_{II} \to 0$, $P_{III} \to 0$, and

$$
P_I = \sum_n \sum_i u_i^{n+1} \frac{\psi_i^n - \psi_i^{n+1}}{\Delta t} \Delta t A_i \to - \int_{\Omega\times R_+} \langle \nu_{(x,t)}, \lambda \rangle \text{sgn}\ (\overline{a} - k)\phi_t(1-\chi_\rho) dx dt.
$$

As $\rho \to 0$, by the dominated convergence theorem, $P_I \to 0$. Using $\phi \nabla \text{sgn}\ (\overline{a} - k) = 0$, we know

that as $h \to 0$, $P_{IV} \to 0$, $P_V \to 0$.

Now we consider

$$\begin{aligned} P_{VI} &= \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} (f(u_T^n) - f(u_i^n))\tau_E a_{ij}^T h_{ij} \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_i - \chi_j) \\ &\quad - \Delta t \sum_n \sum_T \sum_{i<j} (f(u_T^n) - f(u_i^n))\tau_E a_{ij}^T h_{ij} \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n(\chi_{\rho_i} - \chi_{\rho_j}) \\ &\equiv P_a + P_b; \end{aligned}$$

First, by Lemma 3.6, and lemma 4.1, we have $P_b \to 0$ as $\rho \to 0$; second we estimate P_a. Let $\overline{a}_h(x_T, n\Delta t) = \overline{a}_T^n$. Since $|\phi_i^n| \le C$, $|\chi_i - \chi_j| \le 2$, $\sum_{T\in\mathcal{S}_2} 1 \le \frac{C}{h}$, and

$$\begin{aligned} |f(u_i^n) - f(u_T^n)| &\le C|u_i^n - u_T^n| \\ &= C|u_i^n - \overline{a}_i^n + \overline{a}_i^n - \overline{a}_T^n + \overline{a}_T^n - u_T^n| \\ &\le |v_i^n| + |\overline{a}'(\xi)(x_T - x_i)| + |v_T|_{0,\infty,T} \\ &\le 2\|v\|_{0,\infty,T} + h|\overline{a}|_{1,\infty,T}, \end{aligned}$$

we have

$$P_a \le \frac{C}{h}(\|v\|_{0,\infty,\partial S^N} + h|\overline{a}|_{1,\infty,(\Omega\times R_+)})h \to 0 \quad \text{as } h \to 0.$$

Consider

$$G_1 = \Delta t \sum_n \sum_T \int_T f(u^n)\nabla(\text{sgn}\ (\overline{a}^n - k)\phi(\chi - \chi_\rho))dx. \tag{27}$$

By Lemma 6.3, we get

$$\begin{aligned} |G - G_1| &= |\Delta t \sum_n \sum_T \int_T f(u^n)\nabla((\pi - I)\text{sgn}\ (\overline{a}^n - k)\phi(\chi - \chi_\rho))dx| \\ &\le C(h+\rho)\|(I-\pi)\text{sgn}\ (\overline{a} - k)\phi(\chi - \chi_\rho)\|_{1,\infty,\Omega} \\ &= C(h+\rho)\|(I-\pi)\text{sgn}\ (\overline{a} - k)\phi(\chi - \chi_\rho)\|_{1,\infty,T^*} \\ &\le C(h+\rho)\|\chi - \chi_\rho\|_{0,\infty,T^*}\|\phi^n\|_{1,\infty,T^*} \to 0 \quad \text{as } h \to 0,\ \rho \to 0. \end{aligned}$$

Write

$$\begin{aligned} G_1 &= \Delta t \sum_n \sum_T \int_T f(u^n)\nabla\phi^n \text{sgn}\ (\overline{a}^n - k)(\chi - \chi_\rho)dx \\ &\quad + \Delta t \sum_n \sum_T \int_T f(u^n)\phi^n \text{sgn}\ (\overline{a}^n - k)(\nabla\chi - \nabla\chi_\rho)dx \\ &\equiv G_a + G_b. \end{aligned}$$

By (4)–(6),

$$G_a \to \int_{\Omega\times R_+} \langle \nu_{(x,t)}, f(\lambda)\rangle \nabla\phi \text{sgn}\ (\overline{a} - k)(1 - \chi_\rho)dxdt \quad \text{as} \quad h \to 0.$$

As $\rho \to 0$, by the dominated convergence theorem, we get $G_a \to 0$. As $h \to 0$, we have

$$-\Delta t \sum_n \sum_T \int_T f(u^n)\text{sgn }(\overline{a}^n - k)\phi \nabla \chi_\rho dx \to -\int_{\Omega \times R_+} \langle \nu_{(x,t)}, f(\lambda)\rangle \nabla \chi_\rho \phi^n \text{sgn }(\overline{a} - k) dx dt \equiv \mathbf{Ra}.$$

We analyze **Ra** in the following proposition.

Proposition 1:

$$\mathbf{Ra} \to \int_{\Gamma \times R_+} \langle \gamma \nu_{(\overline{x},t)}, f(\lambda)\rangle \text{sgn }(a - k)\phi \cdot n ds dt \quad as \quad \rho \to 0.$$

Proof: We notice (24) and we have $\int_{\frac{\rho}{3}}^{\rho} g'(y)dy = g(\rho) - g(\frac{\rho}{3}) = 1$. By Fubini theorem and intermediate value theorem, we have

$$\begin{aligned}
\mathbf{Ra} &= \int_{\Gamma \times (0,\rho) \times R^+} \langle \nu_{(x(\overline{x},y),t)}, f(\lambda)\rangle \text{sgn }(\overline{a} - k)\phi n(\overline{x}) g'(y) J d\overline{x} dy dt \\
&= \int_{\Gamma \times R_+} \int_0^{\rho} \langle \nu_{(x(\overline{x},y),t)}, f(\lambda)\rangle \text{sgn }(\overline{a} - k)\phi n(\overline{x}) g'(y) J dy d\overline{x} dt \\
&= \int_{\Gamma \times R_+} \langle \nu_{(x(\overline{x},\overline{y}),t)}, f(\lambda)\rangle \text{sgn }(\overline{a} - k)\phi n(\overline{x}) J \int_0^{\rho} g'(y) dy d\overline{x} dt \\
&= \int_{\Gamma \times R_+} \langle \nu_{(x(\overline{x},\overline{y}),t)}, f(\lambda)\rangle \text{sgn }(\overline{a} - k)\phi(\overline{x},\overline{y},t) n(\overline{x}) J(\overline{x},\overline{y}) d\overline{x} dt.
\end{aligned}$$

When $\overline{y} \le y \le \rho \to 0, J(\cdot,\overline{y}) \to J_0$ in $L^1(\Gamma)$ and $J_0 d\overline{x} = ds$, by Lemma 2.2, we get

$$\mathbf{Ra} \to \int_{\Gamma \times R_+} \langle \gamma \nu_{(\overline{x},t)}, f(\lambda)\rangle \text{sgn }(a - k)\phi \cdot n ds dt \quad \text{as} \quad \rho \to 0.$$

□

Let $\chi_\varepsilon = \chi * \rho_{1\varepsilon}$ be the mollification of χ. By Cauchy's inequality, we have as $\varepsilon \le h \to 0$,

$$|\Delta t \sum_n \sum_T \int_T f(u^n)\phi^n \text{sgn }(\overline{a}^n - k)(\nabla \chi - \nabla \chi_\varepsilon) dx| \le C\|\chi - \chi_\varepsilon\|_{1,\Omega} \to 0. \tag{28}$$

By Lemma 6.5, $\|v_h\|_{0,\infty,\partial S^N} \le C(\frac{h+\tau}{\varepsilon_2})^{\frac{1}{2}}$, and by similar proof in Proposition 1, as $\varepsilon \le h \to 0$, we have

$$\Delta t \sum_n \sum_T \int_T \phi f(u^n)\text{sgn }(\overline{a}^n - k)\nabla \chi_\varepsilon dx \to -\int_{\Gamma \times R_+} f(a)\text{sgn }(a - k) \cdot n\phi ds dt.$$

So as $h \to 0$ and $\rho \to 0$,

$$\begin{aligned}
G &\to \int_{\Gamma \times R_+} \langle \gamma \nu_{(\overline{x},t)}, f(\lambda) - f(a)\rangle \text{sgn }(a - k)\phi \cdot n ds dt, \\
P &\to -\int_{\Gamma \times R_+} \langle \gamma \nu_{(\overline{x},t)}, -f(\lambda) + f(a)\rangle \text{sgn }(a - k)\phi \cdot n ds dt.
\end{aligned}$$

Now we set

$$P = \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} \frac{\partial J_2(u_i^n, u_j^n)}{\partial u_j^n}(u_j^n - u_i^n)h_{ij}a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n\{(\chi_i^n - \chi_{\rho_i}^n) - (\chi_j^n - \chi_{\rho_j}^n)\}$$
$$+ \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} \frac{\partial J_1(u_i^n, u_j^n)}{\partial u_j^n}(u_j^n - u_i^n)h_{ij}a_{ij}^T \text{sgn}\ (\overline{a}_i^n - k)\phi_i^n\{-\chi_{\rho_i}^n + \chi_{\rho_j}^n\}$$
$$\equiv B_{II} + P_s.$$

By Lemma 3.5 and Lemma 3.6, as $\rho \to 0$, $P_s \to 0$, and as $h, \rho \to 0$, we have

$$B_{II} \to -\int_{\Gamma\times R_+} \langle \gamma\nu_{(\overline{x},t)}, -f(\lambda) + f(a)\rangle \text{sgn}\ (a-k)\phi \cdot n ds dt.$$

As one more step for the proof of Lemma 6.2, we show the following result.

Proposition 2:

$$|C - C_I| \to 0 \quad as \quad \eta, h \to 0.$$

Proof: By Lemma 6.3, in $C_{I'}$,

$$|C_b| \le C\|(I-\pi)(\phi^n\chi)\|_{1,\infty,\Omega\times R_+} h \le C\|\phi^n\|_{1,\infty,\Omega\times R_+}\|\chi\|_{0,\infty,\Omega} h \to 0 \quad \text{as} \quad h \to 0.$$

Let $C_{II'} \equiv C_{a'} + C_{b'}$, where

$$C_{a'} = -\Delta t \sum_n \sum_T \int_T (f(u^n) - f(k))\text{sg}_\eta(u^n - k)\nabla\phi^n\chi dx, \tag{29}$$
$$C_{b'} = -\Delta t \sum_n \sum_T \int_T (f(u^n) - f(k))\text{sg}_\eta(u^n - k)\nabla\chi\phi^n dx. \tag{30}$$

Then as $\eta, h \to 0$,

$$C_{a'} \to -\int_{\Omega\times R_+} \langle \nu_{(x,t)}, (f(\lambda) - f(k))\text{sgn}(\lambda - k)\rangle \nabla\phi dx dt,$$
$$C_{b'} \to \int_{\Gamma\times R_+} (-f(k) + f(a))\text{sgn}\ (a-k)\phi \cdot n ds dt.$$

$$C - C_I = \Delta t \sum_n \sum_T \sum_{i<j} a_{ij}^T (f(u_T^n) - f(u_i^n)) \cdot \tau_E h_{ij} \text{sg}_\eta(u_i^n - k)((\phi\chi)_i^n - (\phi\chi)_j^n)$$
$$- \Delta t \sum_n \sum_T \sum_{i<j} a_{ij}^T f(u_T^n) \cdot \tau_E h_{ij} (\text{sg}_\eta(u_i^n - k) - \text{sg}_\eta(u_T^n - k))((\phi\chi)_i^n - (\phi\chi)_j^n)$$
$$\equiv C_s + C_t.$$

Since

$$\text{sg}_\eta(u_i^n - k) - \text{sg}_\eta(u_T^n - k) = \text{sg}_\eta' \cdot (-u_T + u_i),$$
$$|u_T - u_i| \le Ch_T|\nabla u^n| \le C|u^n|_{1,T} \le Ch^{-1}|u^n|_{1,1,T},$$

We have, by Lemma 3.6 and Lemma 4.1,

$$|C_t| \le C\Delta t \sum_n \sum_T |u^n|_{1,1,T} \mathrm{sg}'_\eta \le C \int_\Omega |\nabla \mathrm{sg}_\eta(u^n)| dx \to 0 \quad \text{as} \quad \eta \to 0.$$

Let $C_s = C_{sa} + C_{sb}$, where

$$C_{sa} = \Delta t \sum_n \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T (f(u_T^n) - f(u_i^n)) \cdot \tau_E h_{ij} \mathrm{sg}_\eta(u_i^n - k)(\phi_i^n - \phi_j^n),$$
$$C_{sb} = \Delta t \sum_n \sum_{T\in\mathcal{S}_2} \sum_{i<j} a_{ij}^T (f(u_T^n) - f(u_i^n)) \cdot \tau_E h_{ij} \mathrm{sg}_\eta(u_i^n - k)((\phi\chi)_i^n - (\phi\chi)_j^n).$$

Similar to the estimate in A, we have $C_{sa} \to 0$ as $h \to 0$. For C_{sb},

$$|f(u_i^n) - f(u_T^n)| \le C|u_i^n - u_T^n| \le C(2\|v\|_{0,\infty,T} + h|\overline{a}|_{1,\infty,T}),$$
$$|(\phi\chi)_i - (\phi\chi)_j)| \le Ch|\nabla(\phi\chi)| \le C|\phi\chi|_{1,T} \le Ch^{-1}|\phi\chi|_{1,1,T},$$
$$|\nabla\chi|_{0,\infty,T} \le \frac{C}{h}, \quad \sum_{T\in\mathcal{S}_2} \int_T dx \le Ch.$$

We have

$$|C_{sb}| \le C(2\|v\|_{0,\infty,T} + h|\overline{a}|_{1,\infty,T})\Delta t \sum_n \sum_{T\in\mathcal{S}_2} \int_T |\nabla(\phi\chi)| dx \to 0 \quad \text{as} \quad h \to 0,$$

which implies that as $\eta, h \to 0$,$|C - C_I| \to 0$, and

$$C \to -\int_{\Omega\times R_+} \langle \nu_{(x,t)}, (f(\lambda) - f(k))\mathrm{sgn}(\lambda - k)\rangle \nabla\phi dx dt + \int_{\Gamma\times R_+} (-f(k) + f(a))\mathrm{sgn}\ (a-k)\phi \cdot n ds dt.$$

□

Now we prove Lemma 6.2.

Proof of Lemma 6.2: Combining the estimates for the terms A, B, C, D, E, we have

$$\int_{\Omega\times R_+} \langle \nu_{(x,t)}, (\lambda - k)\mathrm{sgn}(\lambda - k)\rangle \cdot \phi_t dx dt + \int_{\Omega\times R_+} \langle \nu_{(x,t)}, (f(\lambda) - f(k))\mathrm{sgn}(\lambda - k)\rangle \nabla\phi dx dt - \int_{\Gamma\times R_+} (f(a) - f(k))\mathrm{sgn}(a-k) \cdot n\phi ds dt + \int_{\Gamma\times R_+} \langle \gamma\nu_{(\overline{x},t)}, -f(\lambda) + f(a)\rangle \mathrm{sgn}\ (a-k)\phi \cdot n ds dt \ge 0.$$

We obtain the estimate with $\phi = \overline{\phi}\overline{\chi}_\xi$,

$$\int_{\Omega\times R_+} (\langle \nu_{(x,t)}, |\lambda - k|\rangle \phi_t + \langle \nu_{(x,t)}, (\mathrm{sgn}\ (\lambda - k)(f(\lambda) - f(k))\rangle \cdot \bigtriangledown\phi) dx dt$$

$$-\int_{\Gamma\times R_+}\langle\gamma\nu_{(\overline{x},t)},f(\lambda)-f(k)\rangle\cdot n(\overline{x})\phi\text{sgn }(a-k)dsdt\geq 0.$$

By the dominated convergence theorem, as $\xi\to 0$, we have

$$\int_{\Omega\times R_+}\langle\nu_{(x,t)},|\lambda-k|\rangle\overline{\phi}_t\overline{\chi}_\xi dxdt+\int_{\Omega\times R_+}\langle\nu_{(x,t)},(f(\lambda)-f(k))\text{sgn }(\lambda-k)\rangle\nabla\overline{\phi}\overline{\chi}_\xi dxdt\to 0.$$

By Lemma 2.2 and the same coordinate change as before, as $\xi\to 0$,

$$\int_{\Omega\times R_+}\langle\nu_{(x,t)},(f(\lambda)-f(k))\text{sgn }(\lambda-k)\rangle\overline{\phi}\nabla\overline{\chi}_\xi dxdt\to$$
$$\int_{\Gamma\times R_+}\langle\gamma\nu_{(\overline{x},t)},(f(\lambda)-f(k))\text{sgn }(\lambda-k)\rangle\overline{\phi}n(\overline{x})dsdt.$$

By the dominated convergence theorem again,

$$\int_{\Gamma\times R_+}\langle\gamma\nu_{(x,t)},(\text{sgn }(\lambda-k)-\text{sgn }(a-k))(f(\lambda)-f(k))\rangle\cdot n\lim_{\xi\to 0}\overline{\chi}_\xi\overline{\phi}dsdt\geq 0.$$

Since $(\text{sgn }(\lambda-\cdot)-\text{sgn }(a(\overline{x},t)-\cdot))(f(\lambda)-f(\cdot))$ is locally Lipschitz continuous on $R\backslash\{a(\overline{x},t)\}$, there is a set $\hat{S}_2$ with Lebesgue measure on $\Gamma\times R_+$ of $\Gamma\times R_+\backslash\hat{S}_2$ equal to zero, such that for $\forall k\in R$, $k\neq a(\overline{x},t)$ a.e. on $\Gamma\times R_+$,

$$\langle\gamma\nu_{(\overline{x},t)},(\text{sgn }(\lambda-k)-\text{sgn }(\overline{a}-k))(f(\lambda)-f(k))\rangle\cdot n(\overline{x})\geq 0.$$

Letting then $k\to a(\overline{x},t)^+_-$, we have proved Lemma 6.2. □

Take $\phi\in C_0^1(\overline{\Omega}\times R_+),\phi\geq 0$. Let $\chi_\xi(x(\overline{x},y))=\begin{cases}0, & y\in[0,\xi],\\ \frac{1}{2}+\frac{3}{4}\frac{y-2\xi}{\xi}-\frac{1}{4}(\frac{y-2\xi}{\xi})^3, & y\in[\xi,3\xi],\\ 1, & y\geq 3\xi.\end{cases}$ where $\overline{x}$, y are defined as before. We write

$$\begin{aligned}&\int_{\Omega\times R_+}\langle\nu_{(x,t)},|\lambda-k|\rangle\phi_t+\langle\nu_{(x,t)},\text{sgn }(\lambda-k))(f(\lambda)-f(k))\rangle\nabla\phi dxdt\\&-\int_{\Gamma\times R_+}\langle\gamma\nu_{(x,t)},f(\lambda)-f(k)\rangle\cdot n\phi\text{sgn }(a-k)dsdt\\=&\int_{\Omega\times R_+}\langle\nu_{(x,t)},|\lambda-k|\rangle(\chi_\xi\phi_t)+\langle\nu_{(x,t)},\text{sgn }(\lambda-k)(f(\lambda)-f(k))\rangle\nabla(\chi_\xi\phi)dxdt\\&+\int_{\Omega\times R_+}\langle\nu_{(x,t)},|\lambda-k|\rangle(1-\chi_\xi)\phi_t+\langle\nu_{(x,t)},\text{sgn }(\lambda-k)(f(\lambda)-f(k))\rangle\nabla((1-\chi_\xi)\phi)dxdt\\&-\int_{\Gamma\times R_+}\langle\gamma\nu_{(x,t)},f(\lambda)-f(k)\rangle\cdot n\phi\text{sgn }(a-k)dsdt\equiv I_\xi+II_\xi+III_\xi.\end{aligned}$$

For $\chi_\xi\phi\in C_0^1(\Omega\times R_+)$, by Lemma 6.1, $I_\xi\geq 0$, and $\nabla((1-\chi_\xi)\phi)=(1-\chi_\xi)\nabla\phi+\phi(-\nabla\chi_\xi)$. As $\xi\to 0$, by the dominated convergence theorem, we have

$$\int_{\Omega\times R_+}\langle\nu_{(x,t)},|\lambda-k|\rangle(1-\chi_\xi)\phi_t+\langle\nu_{(x,t)},\text{sgn }(\lambda-k)(f(\lambda)-f(k))\rangle(1-\chi_\xi)\nabla\phi dxdt\to 0.$$

By Lemma 2.2 and the change of coordinate used as before,

$$II_\xi \to \int_{\Gamma\times R_+} \langle \gamma\nu_{(x,t)}, (f(\lambda)-f(k))\text{sgn }(\lambda-k)\rangle \cdot n\phi ds dt \quad (\xi\to 0).$$

By Lemma 6.2,

$$\lim_{\xi\to 0}(II_\xi + III_\xi) = \int_{\Gamma\times R_+} \langle \gamma\nu_{(x,t)}, (\text{sgn }(\lambda-k)-\text{sgn }(a-k))(f(\lambda)-f(k))\rangle \cdot n\phi ds dt \geq 0.$$

This proves that ν and $\gamma\nu$ satisfy the first part of definition. The fact that ν also satisfies the initial condition, which is proved by weak convergence and the following L_2 stability.

Lemma 6.7. *For $\hat{\phi}\in C_0^1(\Omega)$, we have*

$$\lim_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, \lambda\rangle \hat{\phi} dx = \int_\Omega u_0\hat{\phi} dx, \tag{31}$$

$$\lim_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, |\lambda-u_0|\rangle dx = 0. \tag{32}$$

Proof: Let $\hat{\phi}\in C_0^1(\Omega)$, and take $\hat{\psi}\in C_0^1([0,+\infty)), \hat{\psi}(0)=1$. Let $\hat{\chi}=\hat{\phi}\hat{\psi}$. Multiplying the scheme (10) by $\hat{\chi}_i^n$ and summing up, we get for sufficiently small h,

$$\sum_n\sum_i u_i^{n+1}\frac{\hat{\chi}_i^n-\hat{\chi}_i^{n+1}}{\Delta t}\Delta t A_i + \Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T J_1(u_i^n,u_j^n)h_{ij}(\hat{\chi}_i^n-\hat{\chi}_j^n) = \sum_i u_i^0\hat{\chi}_i^0 A_i.$$

Write

$$\Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T J_1(u_i^n,u_j^n)h_{ij}(\hat{\chi}_i^n-\hat{\chi}_j^n) = -\Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T f(u_i^n)\cdot\tau_E h_{ij}(\tilde{\chi}_i^n-\hat{\chi}_j^n) + R_1,$$

$$-\Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T f(u_i^n)\cdot\tau_E h_{ij}(\hat{\chi}_i^n-\hat{\chi}_j^n) = -\Delta t\sum_n\sum_{T\in\mathcal{S}_1}\int_T f(u^n)\cdot\nabla(\pi\hat{\chi}^n)dx + R_2,$$

where

$$R_1 = \Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T\frac{\partial J_1}{\partial u_j}(u_j^n-u_i^n)h_{ij}(\hat{\chi}_i^n-\hat{\chi}_j^n),$$

$$R_2 = \Delta t\sum_n\sum_{T\in\mathcal{S}_1}\sum_{i<j} a_{ij}^T(f(u_T)-f(u_i^n))\cdot\tau_E h_{ij}(\hat{\chi}_i^n-\hat{\chi}_j^n).$$

Similar to the estimate of A, we prove $R_1\to 0$, $R_2\to 0$ as $h\to 0$.

By (4)–(6),

$$\int_{\Omega\times R_+}\langle\nu_{x,t},\lambda\rangle\hat{\phi}dx\hat{\psi}_t dt + \int_{\Omega\times R_+}\langle\nu_{x,t},f(\lambda)\rangle\cdot\nabla\hat{\phi}dx\hat{\psi}dt + \int_\Omega u_0\hat{\phi}(x,0)dx = 0. \tag{33}$$

We define the functions $\hat{A},\hat{B}\in L^\infty((0,T^*))$ by

$$\hat{A}(t) = \int_\Omega\langle\nu_{(x,t)},\lambda\rangle\hat{\phi}(x)dx, \quad \hat{B}(t) = \int_\Omega\langle\nu_{(x,t)},f(\lambda)\rangle\nabla\hat{\phi}(x)dx.$$

Since $\nu_{(x,t)}$ is a measure-valued solution, $\hat{A}_t + \hat{B} = 0$ in the sense of distributions on R_+. We note $\hat{B} \in L_1((0,T^*))$, which implies $\hat{A}_t \in L_1(0,T^*)$. Hence $\hat{A}(t)$ has bounded variation and $\lim_{t\to 0}\hat{A}(t)$ exists. Take $\hat{\psi}_1 \equiv \hat{\psi}_n = \begin{cases} (1-nt)^2, & t \le \frac{1}{n}, \\ 0, & t > \frac{1}{n}. \end{cases}$ By the dominated convergence theorem,

$$\int_\Omega u_0\hat{\phi}dx = -\lim_{n\to\infty}\int_\Omega \hat{A}(t)(\hat{\psi}_n)_t dt = \lim_{t\to 0}\hat{A}(t).$$

In order to prove (32), we shall use a technique which involves a similar L_2 stability in [9]:

Proposition 3:

$$\int_\Omega \langle \nu_{(x,t)}, \lambda^2\rangle dx \le \int_\Omega u_0^2 dx, \quad \text{for a.e. } t \in (0,T^*).$$

We postpone the proof of Proposition 3 to the end of the section. Assuming first that $u_0 \in C_0^1(\Omega)$. By Proposition 3 and (31):

$$\begin{aligned}\limsup_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, (\lambda-u_0)^2\rangle dx &= \limsup_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, \lambda^2 - u_0^2 - 2u_0(\lambda-u_0)\rangle dx \\ &\le -2\limsup_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, \lambda-u_0\rangle u_0 dx = 0.\end{aligned}$$

Further, by using that Jensen's inequality,

$$\limsup_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, |\lambda-u_0|\rangle dx \le C\limsup_{t\to 0}(\int_\Omega \langle \nu_{(x,t)}, (\lambda-u_0)^2\rangle dx)^{\frac{1}{2}} = 0, \tag{34}$$

which proves the initial condition for regular initial data.

In the more general case $u_0 \in L^\infty(\Omega)$ with supp $u_0 \subset\subset \Omega$, we choose functions $f_n \in C_0^1(\Omega)$, with supp $f_n \subseteq \Omega$ and $\lim_{n\to\infty}\|f_n - u_0\|_{0,\Omega} = 0$. By using Jensen's inequality and Proposition 3, we obtain

$$\begin{aligned}\lim_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, (\lambda-u_0)^2\rangle dx &= \lim_{n\to\infty}\lim_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, (\lambda-f_n)^2\rangle dx = 0, \\ \limsup_{t\to 0}\int_\Omega \langle \nu_{(x,t)}, |\lambda-u_0|\rangle dx &\le C\lim_{t\to 0}(\int_\Omega \langle \nu_{(x,t)}, (\lambda-u_0)^2\rangle dx)^{\frac{1}{2}} = 0.\end{aligned}$$

We now turn to the proof of Proposition 3. Take an arbitrary $\hat{\mathcal{S}}_1 \subset\subset \Omega$ and $\hat{\mathcal{S}}_1$ is a closed polygonal domain, such that supp$u_0 \subset\subset \hat{\mathcal{S}}_1$, $\hat{\mathcal{S}}_1 = \{T \in \mathcal{S}_1 \mid T \bigcap \mathcal{S}_2 = \Phi\}$. Take the characteristic function of $\hat{\mathcal{S}}_1$,

$$\phi_2 = \begin{cases} 1, & x \in \hat{\mathcal{S}}_1, \\ 0, & x \notin \hat{\mathcal{S}}_1. \end{cases}$$

We denote the scheme (10) $u_i^{n+1} \equiv u_i^n + \mathcal{L}(u_i^n, u_j^n)$. Next, we define the sequence $\{w_j^n\}$. For supp$u_0 \subset\subset \hat{\mathcal{S}}_1$, we set

$$\begin{aligned}&\text{Step 1.} \qquad w_i^0 = u_i^0\phi_{2i} \equiv u_i^0. \\ &\text{Step 2.} \quad w_i^{n+1} = (w_i^n + \mathcal{L}(w_i^n, w_j^n))\phi_{2i}, \quad n = 0,1,2,...\end{aligned}$$

By using the value w_i^n on each node, and interpolating linearly on each element T, we extend w_i^n to the whole domain $\overline{\Omega} \times [0, T^*]$ such that it is constant on $[n\Delta t, (n+1)\Delta t)$, $\forall n$, denoted by w_h. We note that

$$\begin{aligned} 0 &= \int_{\Omega_h} \nabla \cdot F(w^n) dx = \int_{\partial\Omega_h} n \cdot F(w^n) ds \\ &= \int_{\Omega_h} w^n \nabla \cdot f(w^n) dx = -\int_{\Omega_h} f(w^n) \cdot \nabla w^n dx \\ &= -\sum_{T\in\mathcal{S}_1} \int_T f(w^n) \cdot \nabla w^n dx = -\sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T f(w_T^n) \cdot \tau_E h_{ij} (w_i^n - w_j^n). \end{aligned}$$

Then we obtain from definition

$$\begin{aligned} w_i^{n+1} &= \{w_i^n + \frac{\Delta t}{A_i} \sum_{T\in\mathcal{T}_{1,i}} \sum_{j\in I_i} a_{i\,j}^T J_1(w_i^n, w_j^n) h_{ij}\} \phi_{2i} \\ &\equiv w_i^n \phi_{2i} + Q_i \phi_{2i}. \end{aligned}$$

Multiplying the equality by $w_{i+1}^n \phi_{2i} A_i$ and we notice

$$\begin{aligned} (w_i^{n+1} - w_i^n \phi_{2i}) w_i^{n+1} \phi_{2i} &= Q_i \phi_{2i} w_i^{n+1} \\ &= Q_i \phi_{2i} (w_i^n \phi_{2i} + Q_i \phi_{2i}) \\ &= Q_i w_i^n \phi_{2i} + Q_i^2 \phi_{2i}. \end{aligned}$$

Summing over i and applying Lemma 3.4 and Lemma 5.1, we have

$$\begin{aligned} &\sum_i Q_i w_i^n \phi_{2i} A_i \\ &= -\Delta t \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T \Big(\frac{e^{\psi_{1j}}(w_j^n - w_i^n)}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} + (f(w_T^n) - f(w_i^n)) \cdot \tau_E + w_i^n O(w_j^n - w_i^n)\Big) \cdot (w_i^n - w_j^n) h_{ij} \\ &\le -C\Delta t h \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T (w_i^n - w_j^n)^2 \le -C\Delta t \delta^{\frac{1}{4}} \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(w_j^n - w_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} h_{ij} \\ &\le -\frac{1}{4}\Delta t \sum_{T\in\mathcal{S}_1} \sum_{i<j} a_{ij}^T \frac{e^{\psi_{1j}}(w_j^n - w_i^n)^2}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} h_{ij}. \end{aligned}$$

Let $p(x) = f(w_i^n)(x - x_i)$, then $\triangle p = 0$. By Lemma 3.3, we have

$$0 = \int_{\Omega_i} \nabla p \cdot \nabla \varphi_i \, dx = \sum_{T\in\mathcal{T}_i} \sum_{j\in I_i} a_{i\,j}^T f(w_i^n) \cdot \tau_E h_{i\,j}.$$

Next by Lemma 3.4, we have

$$\begin{aligned} |Q_i| &= |\frac{\Delta t}{A_i} \{ \sum_{T\in\mathcal{T}_{1,i}} \sum_{j\in I_i} a_{ij}^T \frac{e^{\psi_{1j}}(w_j^n - w_i^n)}{\int_{x_i}^{x_j} \frac{e^{\psi_1}}{\varepsilon_1} ds} h_{ij} + w_i^n O(w_j^n - w_i^n) h_{ij} \}| \\ &\le C \frac{\Delta t}{A_i} \{ \sum_{T\in\mathcal{T}_{1,i}} \sum_{j\in I_i} (h + \varepsilon_1) |a_{ij}^T| |w_j^n - w_i^n| \}, \end{aligned}$$

which implies that

$$\begin{aligned}\sum_i |Q_i|^2\phi_{2i}A_i &\le 2C\sum_i \frac{\Delta t^2}{A_i}\{\sum_{T\in\mathcal{T}_{1,i}}\sum_{j\in I_i}(h+\varepsilon_1)|a_{ij}^T||w_j^n-w_i^n|\}^2\\ &\le 2C\varepsilon_1\Delta t(\delta^{\frac14}h^{\frac12}+2h^{\frac12}+h^{\frac14})\sum_{T\in\mathcal{S}_1}\sum_{i<j}|a_{ij}^T|(w_j^n-w_i^n)^2\\ &\le -\frac14\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(w_j^n-w_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij}.\end{aligned}$$

On the other hand, we have

$$\frac12\sum_i\{(w_i^{n+1})^2-(w_i^n)^2\}\phi_{2i}\le\sum_i(w_i^{n+1}-w_i^n\phi_{2i})w_i^{n+1}\phi_{2i}=\sum_i Q_i\phi_{2i}w_i^{n+1},$$

Combing all above estimates, we obtain

$$\frac12\{\sum_i(w_i^{n+1})^2-\sum_i(w_i^n)^2\}\phi_{2i}A_i-\frac12\Delta t\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(w_j^n-w_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij}\le 0.$$

For a.e. $t\in(0,T^*)$, there exists a positive integral number n_k, such that $n_k\Delta t\le t<(n_k+1)\Delta t$. Summing them up with respect to $n, n=0,1,..n_k-1$, and noting that $w_h(\cdot,t)=w_h(\cdot,n_k\Delta t)$, we get

$$\begin{aligned}\frac12\sum_i(w_i^{n_k})^2\phi_{2i}A_i-\frac12\Delta t\sum_{n=0}^{n_k-1}\sum_{T\in\mathcal{S}_1}\sum_{i<j}a_{ij}^T\frac{e^{\psi_{1j}}(w_j^n-w_i^n)^2}{\int_{x_i}^{x_j}\frac{e^{\psi_1}}{\varepsilon_1}\,ds}h_{ij}&\le\frac12\sum_i(u_i^0)^2\phi_{2i}A_i\\ &\le\frac12\sum_i(u_i^0)^2A_i.\end{aligned}$$

Obviously $\|w_h\|_{L^\infty(\Omega\times R_+)}\le\|u_h\|_{L^\infty(\Omega\times R_+)}$ and using (4)–(6), we obtain Proposition 3. The proof of the initial condition is completed. □

Hence, ν is a mv-solution and by Theorem 2.1 this implies that u_h convergences strongly in $L_1^{loc}(\Omega\times R_+)$ to the unique BV-solution of (1)–(3) as $h\to 0$. We have accomplished the proof of Theorem 3.1.

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Uniform null controllability of the 1-D finite differences space, semi-discretization of the heat equation with locally distributed control

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Abstract. We consider the 1-D finite-difference space semi-discretization of the heat equation with locally distributed control. First, using a result of Russell and Fattorini on biorthogonal series and a seemingly new trigonometrical inequality, we prove the uniform (with respect to the step size) null controllability of this system. Then we show that the sequence of discrete optimal controls strongly converges in a suitable topology to the optimal control of the corresponding continuous model.

Key words. Heat equation, finite differences, space discretization, controllability.

1. Introduction. Our main purpose in this paper is to investigate the uniform null controllability of the finite difference space semi-discretization of the heat equation with locally distributed controls and mixed boundary conditions. Before we get into the heart of the matter, let us say a few words about the existing literature.

Recently, a special attention has been devoted to the study of the boundary observability of the finite difference space semi-discretization of the wave and heat equations (cf. [6, 7, 12, 13, 15]). As far as the wave equation is concerned, it has been observed that the numerical scheme introduces spurious modes at high frequencies which prevents from obtaining uniform observability inequalities. To overcome this obstacle, the authors in [6, 7, 15] use a filtering technique to eliminate the short wave length components of the solutions of the discretized system. This technique was introduced in [4, 5] and its efficiency was highlighted by several numerical experiments. It was also proved in [13] that one might get rid of the filtering technique by choosing analytic initial data. However the situation seems to be completely different in the case of the heat equation; in fact in [12], it is shown that the boundary controllability of the finite difference semi-discretization of the heat equation, with Dirichlet boundary conditions, is uniform without any filtering of high frequency components. To our knowledge, no such result is proved for locally distributed controls. Unlike the case of the boundary control for which the result of [1] leads directly to the uniform discrete observability of the uncontrolled adjoint system, here we still need to prove that each eigenvector of the underlying discrete eigenvalue problem is uniformly (w.r.t. the step size) locally observable (see (A.3)-(A.5) in Appendix). This observation explains our interest in this problem. Besides, our method of proof of the convergence of controls is based on the introduction of extension operators as in [10, 14] while that of [12] amounts to proving the convergence of Fourier coefficients as in [15].

Consider the 1-d heat equation

$$\begin{cases} y_t - y_{xx} = v\chi_\omega \text{ in } (0,1)\times(0,T) \\ y(0,t) = 0, \quad y_x(1,t) = 0 \text{ in } (0,T) \\ y(x,0) = y^0(x) \text{ in } (0,1) \end{cases} \tag{1.1}$$

where ω is a nonempty open subset of $(0,1)$, χ_ω is the characteristic function of ω, and v is the control of minimal $L^2(0,T;L^2(\omega))$-norm such that

$$y(x,T) = 0 \text{ in } (0,1). \tag{1.2}$$

It is well-known that System (1.1) is null controllable for arbitrarily small time $T > 0$; this means that for any initial datum y^0 in $L^2(0,1)$, one can find a control function v which brings the temperature of the system to zero in an arbitrarily short time. This fact is proved in the literature, for all space dimensions, in two different ways:
- by using the Carleman estimates (cf. [2, 3, 9],...)
- by using the biorthogonal series method based on a result of Fattorini and Russell (cf. [1, 12]). This second method is well-adapted to one-dimensional problems since it requires a uniform gap between consecutive eigenvalues of the underlying eigenvalue problem.

One of our objectives in this paper is to find out whether the finite-difference space semi-discretization of (1.1) is uniformly (with respect to the step size) null controllable. To proceed, let N be a positive integer. Set $h = 1/(N+1)$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 < ... < x_N < x_{N+1} = 1,$$

where $x_j = jh$.

The finite-difference space semi-discretization of System (1.1) that we consider is given by

$$\begin{cases} y_j' - \dfrac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = v_j\chi_j \text{ in } (0,T), \quad j = 1,\ 2, ...,\ N \\ y_0(t) = 0, \quad \dfrac{y_{N+1}(t) - y_N(t)}{h} = 0 \text{ in } (0,T) \\ y_j(0) = y_j^0, \quad j = 1,\ 2, ...,\ N \end{cases} \tag{1.3}$$

where y_j^0, χ_j, $j = 1,\ 2, ...,\ N$ are approximations of the functions y^0 and χ_ω respectively. Observe that we do not require that the v_js be approximations of v. As mentioned above, our main goal is to prove that (1.3) is uniformly (with respect to the net-spacing size

parameter) null controllable for arbitrarily small time, and for every $(y_j^0)_j \in \mathbf{R}^N$. As will be shown in the sequel, this amounts to proving that the homogeneous adjoint system associated with (1.3) is uniformly observable. After proving the null controllability result, we will show that the sequence of controls $(v_j)_j$ strongly converges to the control of minimal norm v of System (1.1). This convergence result shows that (1.3) is a good approximation scheme for (1.1). The rest of the paper is organized as follows: in Section 2, we state our main results while Section 3 is devoted to their proofs. Finally, in Appendix, we provide proofs of some estimates used in the proof of Theorem 2.1.

2. Statements of the main results.

THEOREM 2.1. (Controllability). *Let $T > 0$, and $0 < h < 1$. Let $y_h^0 = (y_j^0)_j \in \mathbf{R}^N$. Assume that $\omega = (l_1, l_2)$ with $0 \le l_1 < l_2 \le 1$. Set $l = \texttt{floor}(\frac{l_1}{h})$, and $m = \texttt{ceil}(\frac{l_2}{h})$. Then there exists a unique control $(v_j)_j$ of minimal $L^2(0,T;\mathbf{R}^{m-l+1})$-norm such that the solution of System (1.3) satisfies*

$$y_j(T) = 0, \quad j = 1,\ 2, ...,\ N. \tag{2.1}$$

Moreover the control $(v_j)_j$ satisfies

$$h\sum_{j=l}^{m}\int_0^T |v_j|^2 dt \le Ch\sum_{j=1}^{N}|y_j^0|^2, \tag{2.2}$$

where C is a positive constant independent of h.

If the approximations y_j^0 converge in a suitable topology to y^0, (2.2) tells us that the sequence of controls is uniformly bounded with respect to the net-spacing h. In view of the Hilbert uniqueness method of Lions [11], this result is in contrast with the nonuniform observability results established in the case of wave equations (cf. [6, 7, 13, 15]).

We recall that in the statement of Theorem 2.1, $\texttt{floor}(x)$ denotes the greatest integer less than or equal to x while $\texttt{ceil}(x)$ denotes the smallest integer greater than or equal to x.

Before stating our convergence result, we need some additional notations. Set $y_h = (y_j)_j$, $y_h^0 = (y_j^0)_j$. Introduce the extension operators defined by (see [10]):

$$p_h v_h = \begin{cases} \text{the continuous function, linear in each interval } [jh, (j+1)h], \\ \text{such that } p_h v_h(jh) = v_j,\ j = 0,\ 1,\ ...,\ N+1, \end{cases} \tag{2.3}$$

(2.4)

$$q_h v_h = \begin{cases} \text{the step function defined in each interval } ((j-\frac{1}{2})h, (j+\frac{1}{2})h) \cap (0,1) \\ \text{by } q_h v_h(x) = v_j,\ j = 0,\ 1,\ ...,\ N+1. \end{cases}$$

It is not hard to check that

$$\int_0^1 (p_h v_h)_x (p_h w_h)_x dx = h \sum_{j=0}^{N} \big(\frac{v_{j+1}-v_j}{h}\big)\big(\frac{w_{j+1}-w_j}{h}\big)$$
$$\int_0^1 (q_h v_h)(q_h w_h) dx = h \sum_{j=0}^{N} v_j w_j. \tag{2.5}$$

We are now in the position to state our convergence result:

THEOREM 2.2. (Convergence). *Let y_h denote the solution of (1.3), and let $v_h = (v_j)_j$ be the optimal control. Assume that as h tends to zero,*

$$q_h y_h^0 \to y^0 \text{ strongly in } L^2(0,1) \tag{2.6}$$

where y^0 is the initial datum of (1.1).

Then

$$q_h v_h \to v \text{ strongly in } L^2(0,T;L^2(\omega)), \tag{2.7}$$

where v is the optimal control of System (1.1). *Moreover, we have*

$$\begin{cases} p_h y_h \to y \text{ strongly in } L^2(0,T;H^1(0,1)) \\ q_h y_h \to y \text{ strongly in } L^\infty(0,T;L^2(0,1)) \end{cases} \tag{2.8}$$

where y is the solution of System (1.1).

The convergence hypothesis (2.6) makes sense; indeed with $y_j^0 = \frac{1}{h}\int_{jh}^{(j+1)h} y^0(x)dx$, one can prove that (2.6) holds (cf. [14]). A different convergence approach based on the convergence of Fourier coefficients is presented in [12, 15].

3. Proofs of Theorems 2.1 and 2.2.

3.1. Proof of Theorem 2.1.

This proof essentially relies on the following lemma

LEMMA 3.1. *Let $T > 0$ and $0 < h < 1$. Let $0 \le l < m \le N$. The following assertions are equivalent*:

(i) *There exists a positive constant C_0, independent of h, such that for every $(u_j^0)_j \in \mathbf{R}^N$, one has*

$$h \sum_{j=l}^{m} \int_0^T |u_j|^2 dt \ge C_0 h \sum_{j=1}^{N} |u_j(0)|^2 \tag{3.1}$$

where $(u_j)_j$ is the solution of the system

$$\begin{cases} u_j' + \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0 \text{ in } (0,T), \quad j = 1,\ 2, ...,\ N \\ u_0(t) = 0, \quad \dfrac{u_{N+1}(t) - u_N(t)}{h} = 0 \text{ in } (0,T) \\ u_j(T) = u_j^0, \quad j = 1,\ 2, ...,\ N. \end{cases} \tag{3.2}$$

(ii) *For every $(y_j^0)_j \in \mathbf{R}^N$, there exists a unique control $(v_j)_j$ with minimal $L^2(0,T;\mathbf{R}^{m-l+1})$-norm such that the solution $(y_j)_j$ of (1.3) satisfies*

$$y_j(T) = 0, \quad j = 1,\ 2, ...,\ N \tag{3.3}$$

and the control $(v_j)_j$ satisfies

$$h\sum_{j=l}^{m}\int_0^T |v_j|^2 dt \le \frac{h}{C_0}\sum_{j=1}^{N}|y_j^0|^2. \tag{3.4}$$

where the constant C_0 is the same as above.

Proof of Lemma 3.1. First we assume that (i) holds and prove (ii). To this end, let $J_h : \mathbf{R}^N \to \mathbf{R}$ be the functional defined by

$$J_h((u_j^0)_j) = \frac{h}{2}\sum_{j=l}^{m}\int_0^T |u_j|^2 dt + h\sum_{j=l}^{m} u_j(0)y_j^0. \tag{3.5}$$

It is easy to check that J_h is continuous. On the other hand, thanks to (3.1), J_h is strictly convex and coercive. Therefore, it achieves its minimum value at a unique vector $(z_j^0)_j$ in $\mathbf{R}^N$, and we have the Euler equation

$$h\sum_{j=l}^{m}\int_0^T u_j z_j dt + h\sum_{j=l}^{m} u_j(0)y_j^0 = 0, \forall (u_j)_j, \text{ solution of (3.2)}, \tag{3.6}$$

where $(z_j)_j$ is solution of (3.2) with $z_j(T) = z_j^0$. With (3.6), if we choose $v_j = z_j$ for all j, then we get the control satisfying the claimed conditions. In fact with this choice, multiplying the first equation of (1.3) by hu_j, taking the sum over j and integrating by parts over $[0,T]$, we find

$$h\sum_{j=1}^{N} y_j(T)u_j^0 - h\sum_{j=1}^{N} y_j^0 u_j(0) = h\sum_{j=l}^{m}\int_0^T u_j z_j dt. \tag{3.7}$$

The combination of (3.6) and (3.7) leads to the equation

$$h\sum_{j=1}^{N} y_j(T)u_j^0 = 0, \text{ for all } (u_j^0)_j \in \mathbf{R}^N, \tag{3.8}$$

from which we easily derive $y_j(T) = 0$ for all j. Thanks to (3.1) and (3.6) we also have

$$h\sum_{j=l}^{m}\int_0^T |z_j|^2 dt = -h\sum_{j=l}^{m} z_j(0)y_j^0 \leq \left|h\sum_{j=l}^{m}|z_j(0)|^2\right|^{\frac{1}{2}}\left|h\sum_{j=l}^{m}|y_j^0|^2\right|^{\frac{1}{2}}, \tag{3.9}$$

whence (3.4).

It remains now to show that (ii) implies (i). To this end, multiply the first equation of (1.3) by hu_j, take the sum over j and integrate by parts over $[0,T]$; this operation yields

$$h\sum_{j=l}^{m} u_j(0)y_j^0 = -h\sum_{j=l}^{m}\int_0^T u_j v_j dt. \tag{3.10}$$

Thanks to (3.4), we derive from (3.10) that

$$\left|h\sum_{j=l}^{m} u_j(0)y_j^0\right| \leq h\sum_{j=l}^{m}\int_0^T |u_j v_j| dt \leq C_0^{-\frac{1}{2}}\left|h\sum_{j=l}^{m}\int_0^T |u_j|^2 dt\right|^{\frac{1}{2}}\left|h\sum_{j=l}^{m}|y_j^0|^2\right|^{\frac{1}{2}} \tag{3.11}$$

whence (3.1) and Lemma 3.1 is proved. □

Remark. Lemma 3.1 reduces the proof of Theorem 2.1 to the proof of an inequality of type (3.1) for the solutions $(u_j)_j$ of (3.2). Therefore, we will be done with the proof of Theorem 2.1 if we prove (3.1) for all solutions of (3.2). We have the following result:

PROPOSITION 3.2. *There exists a positive constant C_0 bounded with respect to h such that*

$$h\sum_{j=l}^{m}\int_0^T |u_j|^2 dt \geq C_0 h\sum_{j=1}^{N}|u_j(0)|^2, \quad \forall 0 < h < 1. \tag{3.12}$$

The dependence of C_0 with respect to h will be given in the proof.

To prove Proposition 3.2, we use the Fourier expansion of the solutions of (3.2), and we essentially rely on the following result due to Fattorini and Russell [1]:

LEMMA 3.3. *Let $a_0 > 0$, and let $\nu : \mathbf{R}^+ \to \mathbf{N}$ be a decreasing function satisfying: $\nu(\delta) \to 0$ as $\delta \to 0$. Let $\mathcal{L}(a_0, \nu)$ denote the family of sequences of positive real numbers $(\lambda_n)_n$, $n = 0, 1, 2, ...$ satisfying*

$$\begin{aligned} &\lambda_0 \geq a_0, \quad \lambda_{n+1} - \lambda_n \geq a_0, \quad \forall n = 0, 1, 2, ... \\ &\forall \delta > 0, \quad \sum_{n=\nu(\delta)}^{\infty} \frac{1}{\lambda_n} \leq \delta. \end{aligned} \tag{3.13}$$

Then for all $T > 0$, there exists a constant $C(T)$ depending only on a_0, ν, and T such that

$$\int_0^T \left| \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \right|^2 dt \geq \frac{C(T)}{\sum_{n=0}^{\infty} \frac{1}{\lambda_n}} \sum_{n=0}^{\infty} |c_n|^2 e^{-2\lambda_n T}, \tag{3.14}$$

for all sequences of real numbers $(c_n)_n$.

Proof of Proposition 3.2. We use the Fourier expansion of the solutions to prove this proposition. To this end, we proceed in two steps. First, we state some important results related to the spectral problem associated with System (1.3). Some of these results are elementary and do not need proofs along these lines, the others are not straightforward, and proofs for these are provided in Appendix. Afterwards we use Lemma 3.3 to derive (3.12) and complete the proof.

Step 1. Consider the eigenvalue problem

$$\begin{cases} -\dfrac{X_{j+1} - 2X_j + X_{j-1}}{h^2} = \lambda X_j \\ X_0 = 0, \quad \dfrac{X_{N+1} - X_N}{h} = 0, \quad j = 1,\ 2, ...,\ N. \end{cases} \tag{3.15}$$

Proceeding as in [8], one can show that

$$\begin{aligned} &X_j^{k,h} = \sin\left(\frac{(2k+1)\pi j h}{2-h}\right), \quad j = 0, 1, ..., N \\ &\lambda_{k,h} = \frac{4}{h^2} \sin^2\left(\frac{(2k+1)\pi h}{2(2-h)}\right), \quad k = 0,\ 1, ..., N-1. \end{aligned} \tag{3.16}$$

Moreover, the eigenvectors $X^{k,h}$, $k = 0,\ 1,\ 2,\ ...,\ N-1$ are pairwise orthogonal and

$$\begin{aligned} &h \sum_{j=1}^{N} |X_j^{k,h}|^2 = \frac{2-h}{4}, \quad \forall k,\ h \\ &h \sum_{j=l}^{m} |X_j^{k,h}|^2 \geq \frac{2}{\pi} \left[\frac{h(m-l)\pi}{2-h} - \sin\left(\frac{h(m-l)\pi}{2-h}\right) \right] h \sum_{j=1}^{N} |X_j^{k,h}|^2, \quad \forall k,\ h \end{aligned} \tag{3.17}$$

while the eigenvalues satisfy the following estimates

$$\lambda_{0,h} \geq 1, \quad \lambda_{n+1,h} - \lambda_{n,h} \geq 8, \quad \forall n = 0, 1, 2, ...N-2, \ \forall 0 < h < 1. \tag{3.18}$$

The proofs of (3.17) and (3.18) are provided in Appendix.
Step 2. Any solution of (3.2) may be written as

$$u_j(t) = \sum_{n=0}^{N-1} c_n e^{-\lambda_{n,h}(T-t)} X_j^{n,h}, \text{ with } c_n = \frac{4}{2-h} h \sum_{j=1}^{N} u_j^0 X_j^{n,h}. \tag{3.19}$$

With (3.19), it is easy to check that estimate (3.12) is equivalent to

$$h \sum_{j=l}^{m} \int_0^T |\sum_{n=0}^{N-1} c_n e^{-\lambda_{n,h}(T-t)} X_j^{n,h}|^2 dt \geq C_0 \sum_{n=0}^{N-1} |c_n|^2 e^{-2\lambda_{n,h}T}. \tag{3.20}$$

On the other hand the sequence $(\lambda_{n,h})_n$ may be completed appropriately so as to fulfill all the requirements of Lemma 3.3; indeed it suffices to set $\lambda_{n,h} = [\frac{(2n+1)\pi}{2}]^2$ for $n \geq N$. Therefore we may apply Lemma 3.3; this operation yields

$$\begin{aligned} &h \sum_{j=l}^{m} \int_0^T |\sum_{n=0}^{N-1} c_n e^{-\lambda_{n,h}(T-t)} X_j^{n,h}|^2 dt \\ &\geq \frac{C(T)}{\sum_{n=0}^{\infty} \frac{1}{\lambda_{n,h}}} \sum_{n=0}^{N-1} |c_n|^2 e^{-2\lambda_{n,h}T} h \sum_{j=l}^{m} |X_j^{n,h}|^2. \end{aligned} \tag{3.21}$$

Combining (3.21) with the second line of (3.17), we get (3.20), which completes the proof of Proposition 3.2, and consequently that of Theorem 2.1. □

3.2. Proof of Theorem 2.2. From now on, C denotes different positive constants independent of h.

Using the definitions of p_h and q_h, one easily checks that for every $T \geq t \geq 0$

$$\begin{aligned} &\|q_h y_h(t)\|^2_{L^2(0,1)} + 2\int_0^t \|p_h y_h(s)\|^2_{H^1(0,1)} ds = \|q_h y_h^0\|^2_{L^2(0,1)} \\ &+ 2\int_0^t \int_0^1 q_h v_h(x,s) q_h \chi_h(x) q_h y_h(x,s) ds \end{aligned} \tag{3.22}$$

where $\chi_h = (\chi_j)_j$.

Applying the Gronwall Lemma, and taking into account (2.6) and (3.4) in (3.22) we find, for every $T \geq t \geq 0$

$$\|q_h y_h(t)\|^2_{L^2(0,1)} + 2\int_0^t \|p_h y_h(s)\|^2_{H^1(0,1)} ds \leq C \tag{3.23}$$

which shows that $p_h y_h$ is bounded in $L^2(0,T;H^1(0,1))$, while $q_h y_h$ is bounded in $L^\infty(0,T;L^2(0,1))$. On the other hand, (3.4), (2.6) and the definitions of m and l show that $q_h v_h$ is bounded in $L^2(0,T;L^2(\omega))$. Thus, up to the extraction of a subsequence, we have

$$\begin{cases} p_h y_h \to y \text{ weakly in } L^2(0,T;V) \\ p_h y_h' \to y' \text{ weakly * in } L^2(0,T;V') \\ p_h y_h \to y \text{ strongly in } L^2(0,T;L^2(0,1)) \\ q_h y_h \to y \text{ weakly * in } L^\infty(0,T;L^2(0,1)) \\ q_h v_h \to v \text{ weakly in } L^2(0,T;L^2(\omega)), \end{cases} \tag{3.24}$$

where $V = \{u \in H^1(0,1); u(0) = 0\}$, and V' is its topological dual.

Thanks to (2.5), (3.23), and the definitions of p_h and q_h, we have

$$\int_0^T \int_0^1 |(p_h y_h - q_h y_h)(x,t)|^2 dx = \frac{h^3}{12} \sum_{j=0}^N \int_0^T \Big(\frac{y_{j+1} - y_j}{h}\Big)^2 dt \leq Ch^2, \tag{3.25}$$

so that in (3.24), the implicit claim that the limits of $p_h y_h$ and $q_h y_h$ are the same makes sense.

We have to show now that the limit y is the solution of (1.1), (1.2), the control v being the optimal one. To this end, let $w \in \mathcal{D}([0,1] \times (0,T))$ with $w(0,.) \equiv 0$, and set $w_h = (w_j)_j$ where $w_j = w(jh,.)$. Multiplying the first equation of (1.3) by hw_j, integrating by parts over $(0,T)$ and taking the sum over j, we find

$$\begin{aligned} &- h\sum_{j=1}^N \int_0^T y_j w_j' dt + h\sum_{j=0}^N \int_0^T \Big(\frac{y_{j+1} - y_j}{h}\Big)\Big(\frac{w_{j+1} - w_j}{h}\Big) dt \\ &= h\sum_{j=l}^m \int_0^T v_j w_j dt. \end{aligned} \tag{3.26}$$

Using the definitions of p_h and q_h, it is easy to check that (3.26) is equivalent to

$$\begin{aligned} &- \int_0^T \int_0^1 (q_h y_h)(q_h w_h') dx dt + \int_0^T \int_0^1 (p_h y_h)_x (p_h w_h)_x dx dt \\ &= \int_0^T \int_0^1 (q_h v_h)(q_h \chi_h)(q_h w_h) dx dt. \end{aligned} \tag{3.27}$$

At this stage, we recall the elementary convergence results: For every $w \in \mathcal{D}([0,1]\times(0,T))$

$$\begin{aligned} &p_h w_h \to w \text{ strongly in } L^2(0,T;H^1(0,1)),\\ &q_h w_h \to w \text{ strongly in } L^2(0,T;L^4(0,1)),\\ &q_h\chi_h \to \chi_\omega \text{ strongly in } L^2(0,1),\\ &\text{in particular, } \|q_h\chi_h - \chi_\omega\|_{L^p(0,1)} \le h^{\frac{1}{p}}, \quad \forall 1 \le p < \infty. \end{aligned} \tag{3.28}$$

Thanks to (3.28) and (3.24), we can pass to the limit in all the terms in (3.27) getting

$$-\int_0^T\int_0^1 yw'dxdt + \int_0^T\int_0^1 y_x w_x dxdt = \int_0^T\int_\omega vwdxdt. \tag{3.29}$$

Now, choose w such that we also have $w(1,.) \equiv 0$, then we easily derive the first equation of (1.1) from (3.29). Then choose w with $w(1,.) \not\equiv 0$, it follows that y satisfies the boundary condition at $x = 1$. Thus for y to solve (1.1)-(1.2), it remains to show that $y(0) = y^0$, $y(T) = 0$, and v is the optimal control of (1.1).

First, we show that $y(0) = y^0$ and $y(T) = 0$. For this purpose, let $w \in \mathcal{D}((0,1))$ and $l \in \mathcal{D}([0,T])$, and set $w_h = (w_j)_j$ where $w_j = w(jh)$. Multiplying the first equation of (1.3) by $hw_j l$, integrating by parts over $[0,T]$ and taking the sum over j, we find

$$\begin{aligned} &-h\sum_{j=1}^N y_j^0 w_j l(0) - h\sum_{j=1}^N\int_0^T y_j w_j l'dt + h\sum_{j=0}^N\int_0^T \Big(\frac{y_{j+1}-y_j}{h}\Big)\Big(\frac{w_{j+1}-w_j}{h}\Big)ldt\\ &= h\sum_{j=l}^m\int_0^T v_j w_j dt. \end{aligned} \tag{3.30}$$

Using the definitions of p_h and q_h once more, it is easy to check that (3.30) is equivalent to

$$\begin{aligned} &-l(0)\int_0^1 (q_h y_h^0)(q_h w_h)dx - \int_0^T\int_0^1 (q_h y_h)(q_h w_h)l'dxdt\\ &+\int_0^T\int_0^1 (p_h y_h)_x (p_h w_h)_x l dxdt = \int_0^T\int_0^1 (q_h v_h)(q_h\chi_h)(q_h w_h)ldxdt. \end{aligned} \tag{3.31}$$

Passing to the limit as $h \to 0$ in (3.31), we get

$$-l(0)\int_0^1 y^0 w dx - \int_0^T\int_0^1 ywl'dxdt + \int_0^T\int_0^1 y_x w_x l dxdt = \int_0^T\int_\omega vwldxdt \tag{3.32}$$

from which we easily derive $y(0) = y^0$ and $y(T) = 0$. Thus v is a control for System (1.1). It remains to prove that the sequences $(q_h y_h)$, $(p_h y_h)$, $(q_h v_h)$ strongly converge in their respective spaces, and that v is the optimal control of (1.1). This will show in particular that the whole sequence $(q_h v_h)$, not only a subsequence, converges, from which we will derive (2.7)-(2.8) and complete the proof of Theorem 2.2. To this end, we proceed in steps, and *we assume that the sequence* $\{h\}$ *denotes the subsequence extracted above.*

Step 1. (Strong convergence of controls). Let $\tau \in (0, T)$. Proceeding as in the proof of Proposition 3.2, one can show that

$$||q_h v_h(\tau)||_{L^2((0,1))} \leq C(T - \tau), \tag{3.33}$$

where $C(T - \tau)$ is independent of h.

Consequently, there exists some $v^0 \in L^2((0, 1))$ such that, up to a subsequence,

$$q_h v_h(\tau) \to v^0 \text{ weakly in } L^2((0, 1)). \tag{3.34}$$

It follows from (3.34) that

$$\begin{aligned} &q_h v_h \to v \text{ weakly * in } L^\infty(0, \tau; L^2((0, 1))) \\ &p_h v_h \to v \text{ weakly in } L^2(0, \tau; V) \\ &(p_h v_h)_t \to v_t \text{ weakly * in } L^2(0, \tau; V') \end{aligned} \tag{3.35}$$

where v is the same as in (3.24), (remember our remark about the sequence $\{h\}$ just before this step). Accordingly,

$$q_h v_h \to v \text{ strongly in } L^2(0, \tau; L^2(\omega)), \forall 0 < \tau < T. \tag{3.36}$$

Hence

$$q_h v_h \to v \text{ strongly in } L^2(0, T; L^2(\omega)). \tag{3.37}$$

Before showing that v is the optimal control for System (1.1), it is worth noting that

$$q_h v_h(0) \to v(0) \text{ weakly in } L^2((0, 1)). \tag{3.38}$$

Step 2. (v is the optimal control of (1.1)). By optimal control of (1.1), we mean a control function v which minimizes the quantity $\int_0^T \int_\omega |v|^2 dxdt$ among the admissible controls. To show this, it is enough to prove that v satisfies

$$\int_0^T \int_\omega |v|^2 dxdt + \int_0^1 v(0) y^0 dx = 0, \tag{3.39}$$

and the Euler equation

$$\int_0^T \int_\omega vudxdt + \int_0^1 u(0)y^0 dx = 0, \tag{3.40}$$

for all u solution of

$$\begin{cases} u_t + u_{xx} = 0 \text{ in } (0,1)\times(0,T) \\ u(0,t) = 0, \quad u_x(1,t) = 0 \\ u(x,T) = u^0(x) \text{ in } (0,1), \end{cases} \tag{3.41}$$

where $u^0 \in L^2((0,1))$.

The equation (3.39) follows from (3.6), (3.28), (2.6), (3.37), and (3.38). It remains to show that (3.40) holds. To this end, let u^0 be an arbitrary element of $L^2((0,1))$. Assuming that $u_j^0 = \frac{1}{h}\int_{jh}^{(j+1)h} u^0(x)dx$ in (3.2), one can show that

$$\begin{aligned} &q_h u_h^0 \to u^0 \text{ strongly in } L^2((0,1)) \\ &q_h u_h(0) \to u(0) \text{ weakly in } L^2((0,1)) \\ &q_h u_h \to u \text{ strongly in } L^2(0,T;L^2(\omega)), \end{aligned} \tag{3.42}$$

where u solves (3.41).

Now we know that the discrete optimal control $(v_j)_j$ satisfies (3.6) so that

$$\int_0^T \int_0^1 q_h v_h q_h \chi_h q_h u_h dxdt + \int_0^1 q_h u_h(0) q_h y_h^0 dx = 0. \tag{3.43}$$

Passing to the limit in (3.43) we get (3.40). Since the optimal control for (1.1) is unique, it follows that the whole sequence $(q_h v_h)$ converges to v.

Step 3. (Strong convergence of states). We shall now prove (2.8). We already have the corresponding weak convergences; if we can prove the convergence of norms, then we will be done. First we show that $||p_h y_h||_{L^2(0,T;V)} \to ||y||_{L^2(0,T;V)}$. Since $q_h y_h(T) = 0$, it follows from (3.22) that

$$\begin{aligned} 2\int_0^T \|p_h y_h(s)\|_V^2 ds &= \|q_h y_h^0\|_{L^2(0,1)}^2 \\ &+ 2\int_0^T \int_0^1 q_h v_h(x,s) q_h \chi_h(x) q_h y_h(x,s) ds, \end{aligned} \tag{3.44}$$

so that using (2.6), (3.28), (3.37) and passing to the limit in (3.44), one finds

$$\begin{aligned}\lim_{h\to 0} 2\int_0^T \|p_h y_h(s)\|^2_{H^1(0,1)} ds &= \|y^0\|^2_{L^2(0,1)} + 2\int_0^T \int_\omega v(x,s)y(x,s)ds \\ &= 2\int_0^T \|y(s)\|^2_V ds, \text{ (since } y \text{ satisfies (1.2).}\end{aligned} \tag{3.45}$$

Therefore $p_h y_h \to y$ stronly in $L^2(0,T;V)$ as claimed. Let us now show that $||q_h y_h||_{L^\infty(0,T;L^2(0,1))} \to ||y||_{L^\infty(0,T;L^2(0,1))}$. Thanks to (3.22) and (3.45) we have, for $0 < t < T$

$$\|q_h y_h(t)\|_{L^2(0,1)} \to ||y(t)||_{L^2(0,1)},$$

which coupled with (3.24) completes the proof of Theorem 2.2. □

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Appendix. Proofs of (3.17) and (3.18).

Proof of (3.17.1). We have

$$
\begin{aligned}
h\sum_{j=0}^{N}|X_j^{k,h}|^2 &= h\sum_{j=0}^{N}|\sin(\frac{(2k+1)\pi jh}{2-h})|^2\\
&= \frac{1}{2}-\frac{h}{2}\sum_{j=0}^{N}\cos(\frac{2(2k+1)\pi jh}{2-h}),\ (\text{since } h(N+1)=1)\\
&= \frac{1}{2}-\frac{h}{2}\frac{\cos(\frac{(2k+1)\pi Nh}{2-h})\sin(\frac{(2k+1)\pi(N+1)h}{2-h})}{\sin(\frac{(2k+1)\pi h}{2-h})} \qquad (A.1)\\
&= \frac{1}{2}-\frac{h}{4}\frac{(\sin(\frac{(2k+1)\pi(2N+1)h}{2-h})+\sin(\frac{(2k+1)\pi h}{2-h}))}{\sin(\frac{(2k+1)\pi h}{2-h})}\\
&= \frac{2-h}{4},\quad \forall k,\ h.
\end{aligned}
$$

Proof of (3.17.2). For this proof, we will use the elementary trigonometric inequalities

$$
\begin{aligned}
&|\sin(nx)|\le n|\sin(x)|,\ \text{for all nonnegative integer } n,\ \text{and all } x\\
&\qquad (A.2)\\
&\frac{x}{\tan x}<1 \text{ for all } x\in(0,\pi/2).
\end{aligned}
$$

The first -from top- of these inequalities seems to be new, though it is simple and can be easily proved by an induction argument. We now turn to the proof of (3.17.2)

$$
\begin{aligned}
h\sum_{j=l}^{m}|X_j^{n,h}|^2 &= h\sum_{j=l}^{m}|\sin(\frac{(2n+1)\pi jh}{2-h})|^2\\
&= \frac{(m-l+1)h}{2}-\frac{h}{2}\sum_{j=l}^{m}\cos(\frac{2(2n+1)\pi jh}{2-h})\\
&= \frac{(m-l+1)h}{2}-\frac{h}{2}\frac{\cos(\frac{(2n+1)(m+l)\pi h}{(2-h)})\sin(\frac{(2n+1)(m-l+1)\pi h}{(2-h)})}{\sin(\frac{(2n+1)\pi h}{(2-h)})}\\
&= \frac{(m-l+1)h}{2}-\frac{h}{2}\frac{\cos(\frac{(2n+1)(m+l)\pi h}{(2-h)})\sin(\frac{(2n+1)(m-l)\pi h}{(2-h)})}{tan(\frac{(2n+1)\pi h}{(2-h)})}\\
&-\frac{h}{2}\cos(\frac{(2n+1)(m+l)\pi h}{(2-h)})\cos(\frac{(2n+1)(m-l)\pi h}{(2-h)})\\
&\geq \frac{(m-l+1)h}{2}-\frac{h}{2}\frac{|\sin(\frac{(2n+1)(m-l)\pi h}{(2-h)})|}{|tan(\frac{(2n+1)\pi h}{(2-h)})|}-\frac{h}{2}\\
&\geq \frac{(m-l)h}{2}-\frac{h(2n+1)}{2}\frac{\sin(\frac{(m-l)\pi h}{(2-h)})}{|tan(\frac{(2n+1)\pi h}{(2-h)})|},
\end{aligned}
\tag{A.3}
$$

by applying the first (from top) inequality of (A.2).

At this stage, we observe that if $\frac{(2n+1)h}{(2-h)}<\frac{1}{2}$, then $\frac{(2n+1)\pi h}{(2-h)}<\frac{\pi}{2}$ so that we may be able to apply the second inequality of (A.2). From now on, we proceed in steps.

Step 1. We assume $\frac{(2n+1)h}{(2-h)}<\frac{1}{2}$. It then follows from (A.3) that

$$
\begin{aligned}
h\sum_{j=l}^{m}|X_j^{n,h}|^2 &= h\sum_{j=l}^{m}|\sin(\frac{(2n+1)\pi jh}{2-h})|^2\\
&\geq \frac{(m-l)h}{2}-\frac{(2-h)}{2\pi}\frac{h(2n+1)\pi}{(2-h)tan(\frac{(2n+1)\pi h}{(2-h)})}\sin(\frac{(m-l)\pi h}{(2-h)})\\
&\geq \frac{(m-l)h}{2}-\frac{(2-h)}{2\pi}\sin(\frac{(m-l)\pi h}{(2-h)}),
\end{aligned}
\tag{A.4}
$$

by applying the second (from top) inequality of (A.2).

Step 2. We assume $\frac{(2n+1)h}{(2-h)}>\frac{1}{2}$. An elementary algebra shows that equality never holds, so that this is the last step in our proof. We will reduce this case to the preceding

one. To proceed, set $p_n = N-1-n$. It is easy to check that $\frac{(2p_n+2)h}{(2-h)} = 1-\frac{(2n+1)h}{(2-h)} < 1/2$. Using these relations in (A.3), we get

$$
\begin{aligned}
h\sum_{j=l}^{m}|X_j^{n,h}|^2 &= h\sum_{j=l}^{m}|\sin(\frac{(2n+1)\pi jh}{2-h})|^2\\
&\geq \frac{(m-l+1)h}{2} - \frac{h}{2}\frac{|\sin(\frac{(2p_n+2)(m-l)\pi h}{(2-h)})|}{|tan(\frac{(2p_n+2)\pi h}{(2-h)})|} - \frac{h}{2} \qquad (A.5)\\
&\geq \frac{(m-l)h}{2} - \frac{h(2p_n+2)}{2}\frac{\sin(\frac{(m-l)\pi h}{(2-h)})}{|tan(\frac{(2p_n+2)\pi h}{(2-h)})|}.
\end{aligned}
$$

From this point we proceed as in Step 1, since $\frac{(2p_n+2)h}{2-h} < 1/2$. This completes the proof of (3.17).

Let us turn now to the proof of (3.18). To prove the estimates, we use the fact that

$$\sin x \geq 2x/\pi, \text{ for all } x \in [0,\pi/2]. \qquad (A.6)$$

We have for all n

$$\lambda_{n,h} = \frac{4}{h^2}\sin^2(\frac{(2n+1)\pi h}{2(2-h)}) \geq \frac{4}{h^2}(\frac{2}{\pi}(\frac{(2n+1)\pi h}{2(2-h)}))^2 = 4(2n+1)^2/(2-h)^2 \qquad (A.7)$$

from which we derive $\lambda_{0,h} \geq 1$. It remains to check the uniform gap condition. For $n = 0,\ 1,\ 2,\ ...,\ N-2$, we have

$$
\begin{aligned}
\lambda_{n+1,h} - \lambda_{n,h} &= \frac{4}{h^2}\left[\sin^2(\frac{(2n+3)\pi h}{2(2-h)}) - \sin^2(\frac{(2n+1)\pi h}{2(2-h)})\right]\\
&= \frac{2}{h^2}\left[\cos(\frac{(2n+1)\pi h}{2-h}) - \cos(\frac{(2n+3)\pi h}{2-h})\right] \qquad (A.8)\\
&= \frac{4}{h^2}\sin(\frac{\pi h}{2-h})\sin(\frac{(2n+2)\pi h}{2-h})
\end{aligned}
$$

At this stage, we observe that for n large enough, $\frac{(2n+2)\pi h}{(2-h)}$ may be greater than $\pi/2$ thus precluding us from using (A.2) to conclude. So we proceed in steps.

Step 1. Assume that $\frac{(2n+2)h}{(2-h)} \leq \frac{1}{2}$. It follows from $(A.8)$ that

$$
\begin{aligned}
\lambda_{n+1,h} - \lambda_{n,h} &= \frac{4}{h^2}\sin(\frac{\pi h}{2-h})\sin(\frac{(2n+2)\pi h}{2-h})\\
&\geq \frac{4}{h^2}\frac{2}{\pi}\frac{\pi h}{(2-h)}\frac{2}{\pi}\frac{(2n+2)\pi h}{(2-h)} \qquad (A.9)\\
&\geq \frac{32(n+1)}{(2-h)^2}\\
&\geq 8.
\end{aligned}
$$

Step 2. Assume now that $\frac{(2n+2)h}{(2-h)} > \frac{1}{2}$. We proceed as in Step 2 of the proof of (3.17.2). Set $p_n = N-2-n$. It follows that $\frac{(2p_n+3)h}{(2-h)} = 1 - \frac{(2n+2)h}{(2-h)} < \frac{1}{2}$. Using these relations and the identity $\sin(\pi - x) = \sin x$ in $(A.8)$, we find

$$
\begin{aligned}
\lambda_{n+1,h} - \lambda_{n,h} &= \frac{4}{h^2}\sin(\frac{\pi h}{2-h})\sin(\frac{(2n+2)\pi h}{2-h}) \\
&= \frac{4}{h^2}\sin(\frac{\pi h}{2-h})\sin(\frac{(2p_n+3)\pi h}{2-h}) \\
&\geq \frac{4}{h^2}\frac{2}{\pi}\frac{\pi h}{(2-h)}\frac{2}{\pi}\frac{(2p_n+3)\pi h}{(2-h)} \\
&\geq \frac{32(p_n+1)}{(2-h)^2} \\
&\geq 8.
\end{aligned} \tag{A.10}
$$

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Two Sharp Inequalities and Applications

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ABSTRACT. Two sharp inequalities are established. Upper and lower error bounds for the well-known Simpson's quadrature rule are obtained. Applications in numerical integration are also given.

Key words. Simpson's quadrature rule, error bounds, sharp inequalities, perturbations, numerical integration.
MSC: 26D10, 41A55, 65D30.

1 Introduction

In recent years a number of authors have written about generalizations of Ostrowski's inequality. Ostrowski's inequality gives an error bound for the following simple quadrature rule:

$$\int_a^b f(t)dt = f(x)(b-a) + R(f;a,b,x), \tag{1}$$

where $x \in [a,b]$. The mentioned generalizations often lead to estimates of errors for some known and some new quadrature rules. (They have many other uses.)

In this paper we consider the following 3-point quadrature rule:

$$\int_a^b f(t)dt = \frac{1}{4}\left[f(a) + 2f(\frac{a+b}{2}) + f(b)\right](b-a) + R(f;a,b) \tag{2}$$

and the well-known Simpson's quadrature rule:

$$\int_a^b f(t)dt = \frac{1}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right](b-a) + R(f;a,b). \tag{3}$$

In [9] it is shown that (2) has a better estimate of error than (3) when these estimates are expressed in terms of first derivatives. Here we consider perturbations of (2) and (3). Similar perturbations for the mid-point and trapezoid rules are considered in [5] and [6]. A perturbed Simpson's rule is also considered in [13].

Let us additionally mention that Simpson's inequality is considered in [8], [9] and [13], while some inequalities for (2) are derived in [9] and [13].

Furthermore, we give upper and lower error bounds for the above quadrature rules. In Section 3 we give applications in numerical integration.

2 Main results

We begin with general considerations and observations.

Let $g : [a,b] \to R$ be an absolutely continuous function. Let γ, Γ be real numbers such that $\gamma \leq g'(t) \leq \Gamma$, $t \in [a,b]$ $(a.e)$. If $g'(t_0)$ does not exist, for some $t_0 \in [a,b]$, then we set $g'(t_0) = \frac{\Gamma+\gamma}{2}$, by definition. This restriction does not affect to validity of the results obtained in this paper. That is, we consider such types of problems that the above restriction has no practical importance. It is only important from a theoretical point of view.

We now describe a general setting from which we derive all further results.

If we have a Peano kernel $p_k(t)$ then $R(f) = \int_a^b p_k(t) f^{(k)}(t)dt$ is a remainder term (error) of a corresponding quadrature formula $Q(f)$. We have $\int_a^b f(t)dt = Q(f) + R(f)$. The usual Peano error bound is given by

$$|R(f)| \leq \left\| f^{(k)} \right\|_\infty \int_a^b |p_k(t)|\, dt. \tag{4}$$

In recent time it is shown that many improvements of the estimation (4) can be obtained if we replace $p_k(t) f^{(k)}(t)$ with $[p_k(t) - C_1]\, f^{(k)}(t)$ or $p_k(t) \left[f^{(k)}(t) - C_2\right]$ or $[p_k(t) - C_3] \left[f^{(k)}(t) - C_4\right]$, where C_i, $i = 1,2,3,4$, are constants. For example, in [7] the author choose $C_1 = \frac{1}{b-a} \int_a^b p_k(t)dt$. In [13] the authors choose $C_2 = \frac{1}{b-a} \int_a^b f^{(k)}(t)dt$. Such perturbations lead to inequalities of Ostrowski (Ostrowski-Grüss, Ostrowski-Chebyshev, etc.) type.

In such a way we also derive new quadrature formulas and perturbations of known quadrature formulas. The best possible results are obtained if we can prove that the error bounds are sharp.

Here we choose $C_2 = \frac{\Gamma_k+\gamma_k}{2}$, where γ_k,Γ_k are real numbers such that $\gamma_k \leq f^{(k)}(t) \leq \Gamma_k$, $t \in [a,b]$. Specially, if $f^{(k)}$ is a continuous function and

$$\gamma_k = \min_{t\in[a,b]} f^{(k)}(t), \quad \Gamma_k = \max_{t\in[a,b]} f^{(k)}(t) \tag{5}$$

then $P(t) = \frac{\Gamma_k+\gamma_k}{2}$ is a polynomial of best uniform approximation and we have

$$\left\| f^{(k)} - P \right\|_\infty = \frac{\Gamma_k - \gamma_k}{2}.$$

Thus, such a choice is a natural choice. It causes a perturbation in the original quadrature formula (obtained from $\int_a^b p_k(t) f^{(k)}(t)dt$). The main consequences of such a choice are:

(i) error bounds of perturbed formulas are better than error bounds of original formulas (see Remarks 4 and 11),

(ii) error bounds of perturbed formulas are sharp (see Theorems 3 and 10),

(iii) corresponding composite quadrature formulas have only one additional term with respect to original composite formulas (see Theorems 15 and 17),

(iv) error bounds of corresponding composite formulas are better than error bounds of original composite formulas (a consequence of (i)),

(v) degrees of precision of the perturbed formulas are higher than degrees of precision of the original formulas (see Remarks 4 and 11).

We also choose $C_2 = \gamma_k$ and $C_2 = \Gamma_k$. Sometimes these choices give better error bounds than the choice $C_2 = \frac{\Gamma_k+\gamma_k}{2}$ (see Remarks 6 and 13).

We now define two finite sequences of harmonic (Appell-like) polynomials:

$$\begin{array}{cc} P_0(t) = 1 & Q_0(t) = 1 \\ P_1(t) = t - \frac{a+\alpha(x)}{2} & Q_1(t) = t - \frac{b+\beta(x)}{2} \\ P_2(t) = \frac{1}{2}(t-a)(t-\alpha(x)) & Q_2(t) = \frac{1}{2}(t-b)(t-\beta(x)) \end{array}$$

where $x \in [a,b]$ and $\alpha(x)$, $\beta(x)$ depend on x. We also define the functions:

$$S_k(t) = \begin{cases} P_k(t), & a \le t \le x \\ Q_k(t), & x < t \le b, \end{cases} \tag{6}$$

for $k = 0, 1, 2$. Additionally, we need the following functions:

$$I_1(f;a,b,\alpha,\beta,x) = \frac{\alpha(x)-a}{2} f(a) + \frac{b+\beta(x)-a-\alpha(x)}{2} f(x) + \frac{b-\beta(x)}{2} f(b), \tag{7}$$

$$I_2(f;a,b,\alpha,\beta,x) = -\frac{1}{2} f'(x) \left[(x-a)(x-\alpha(x)) - (x-b)(x-\beta(x)) \right], \tag{8}$$

$$I_3(f;a,b,\alpha,\beta,x) = \frac{(x-a)^3 + (b-x)^3}{6} + \frac{1}{4} \left[(x-a)^2(a-\alpha(x)) - (x-b)^2(b-\beta(x)) \right]. \tag{9}$$

Lemma 1 *Let S_k, $k = 0, 1, 2$ and I_j, $j = 1, 2, 3$ be defined by (6)–(9). If $f' : [a,b] \to R$ is an absolutely continuous function then we have*

$$\int_a^b f(t)dt = I_1 + I_2 + CI_3 + R(f) \tag{10}$$

where C is a constant, $I_j = I_j(f;a,b,\alpha,\beta,x)$, $j=1,2,3$ and

$$R(f) = \int_a^b [f''(t) - C]\, S_2(t)dt, \tag{11}$$

$$|R(f)| \leq \sup_{t\in[a,b]} |f''(t) - C| \int_a^b |S_2(t)|\, dt. \tag{12}$$

Proof. Integrating by parts, we obtain

$$\int_a^b S_2(t)f''(t)dt = -I_2 - \int_a^b S_1(t)f'(t)dt = -I_1 - I_2 + \int_a^b f(t)dt.$$

We also have

$$\int_a^b S_2(t)dt = I_3.$$

From the above two relations we see that (10)-(11) hold. The estimation (12) is obvious. ■

Remark 2 *The results of Lemma 1 can be generalized in a way given in [2] or [3]. Such a generalization leads to a summation formula for approximate determining of definite integrals with a corresponding error bound. Here we give different approach to the same problem. Namely, we derive a perturbed quadrature formula (from Lemma 1) and give a corresponding composite formula. In fact, the main goal is to obtain error bounds for these formulas.*

Theorem 3 *Under the assumptions of Lemma 1 suppose that $\gamma \leq f''(t) \leq \Gamma$, $t\in[a,b]$, where γ,Γ are real numbers. Then we have*

$$\left| \int_a^b f(t)dt - \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + \frac{\Gamma+\gamma}{96}(b-a)^3 \right| \leq \frac{\Gamma-\gamma}{96}(b-a)^3. \tag{13}$$

The inequality (13) is sharp.

Proof. If we choose $\alpha(x) = \beta(x) = x = \frac{a+b}{2}$ then we have

$$I_1(f;a,b,\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2}) = \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a), \tag{14}$$

$$I_2(f;a,b,\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2}) = 0, \tag{15}$$

$$I_3(f;a,b,\frac{a+b}{2},\frac{a+b}{2},\frac{a+b}{2}) = -\frac{(b-a)^3}{48} \tag{16}$$

and

$$\int_a^b |S_2(t)|\,dt = \frac{(b-a)^3}{48}. \tag{17}$$

From (14)–(16) and (10) with $C = \frac{\Gamma+\gamma}{2}$ it follows

$$\begin{aligned} &\int_a^b f(t)dt - \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + \frac{\Gamma+\gamma}{96}(b-a)^3 \qquad (18)\\ = &\int_a^b \left[f''(t) - \frac{\Gamma+\gamma}{2}\right] S_2(t)dt. \end{aligned}$$

From (12), (17), (18) and $\sup_{t\in[a,b]} \left|f''(t) - \frac{\Gamma+\gamma}{2}\right| \leq \frac{\Gamma-\gamma}{2}$ we get (13).

We now show that (13) is sharp. Let us define the function

$$f(t) = \left|t - \frac{a+b}{2}\right|^r,\ r > 2.$$

Then we have

$$f'(t) = r\left|t - \frac{a+b}{2}\right|^{r-1} sgn(t - \frac{a+b}{2}),$$

$$f''(t) = r(r-1)\left|t - \frac{a+b}{2}\right|^{r-2} sgn(t - \frac{a+b}{2}),$$

$$\gamma = -r(r-1)\frac{(b-a)^{r-2}}{2^{r-2}},\ \Gamma = r(r-1)\frac{(b-a)^{r-2}}{2^{r-2}}$$

and

$$\int_a^b f(t)dt = \frac{(b-a)^{r+1}}{(r+1)2^r}.$$

The left-hand side of (13) becomes:

$$L.H.S.(13) = \frac{(b-a)^{r+1}}{2^r}\left|\frac{1}{r+1} - \frac{1}{2}\right|.$$

The right-hand side of (13) becomes:

$$R.H.S.(13) = \frac{1}{3}\frac{r(r-1)}{2^{r+2}}(b-a)^{r+1}.$$

We have

$$\lim_{r\to 2} L.H.S.(13) = \frac{(b-a)^3}{24}$$

and

$$\lim_{r\to 2} R.H.S.(13) = \frac{(b-a)^3}{24}.$$

Hence,

$$\lim_{r\to 2} L.H.S.(13) = \lim_{r\to 2} R.H.S.(13).$$

Thus, (13) is sharp. ■

Remark 4 *In the above theorem a perturbation of the averaged midpoint-trapezoidal quadrature rule*

$$\int_a^b f(t)dt = \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + R(f) \tag{19}$$

is considered. Since $|R(f)| \leq \frac{\|f''\|_\infty}{48}(b-a)^3$ *it is not difficult to see that the perturbed rule*

$$\int_a^b f(t)dt = \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) - \frac{\Gamma+\gamma}{96}(b-a)^3 + R_1(f) \tag{20}$$

has a better estimation of error. Further, the rule (19) is exact for polynomials of degree ≤ 1*, while the perturbed rule (20) is exact for polynomials of degree* ≤ 3*, if we choose* γ, Γ *as in (5) (*$k=2$*,* $\gamma=\gamma_2$*,* $\Gamma=\Gamma_2$*).*

Corollary 5 *Under the assumptions of Theorem 3 we have*

$$\left| \int_a^b f(t)dt - \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + \frac{\gamma}{48}(b-a)^3 \right| \leq \frac{S-\gamma}{32}(b-a)^3 \tag{21}$$

and

$$\left| \int_a^b f(t)dt - \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + \frac{\Gamma}{48}(b-a)^3 \right| \leq \frac{\Gamma-S}{32}(b-a)^3, \tag{22}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. We choose $\alpha(x) = \beta(x) = x = \frac{a+b}{2}$. Then we have

$$\int_a^b S_2(t)\left[f''(t)-\gamma\right]dt \tag{23}$$

$$= \int_a^b f(t)dt - \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) + \frac{\gamma}{48}(b-a)^3,$$

$$\left| \int_a^b S_2(t)\,[f''(t)-\gamma]\,dt \right| \le \max_{t\in[a,b]} |S_2(t)| \int_a^b |f''(t)-\gamma|\,dt, \tag{24}$$

$$\max_{t\in[a,b]} |S_2(t)| = \frac{(b-a)^3}{32} \tag{25}$$

and

$$\begin{aligned} \int_a^b |f''(t)-\gamma|\,dt &= f'(b)-f'(a)-\gamma(b-a) \\ &= (S-\gamma)(b-a). \end{aligned} \tag{26}$$

From (23)-(26) we easily get (21).

In a similar way we can prove that (22) holds. ■

Remark 6 *The above obtained estimates can be better than the estimate obtained in Theorem 3. For example, if we consider the function* $f(x)=\exp(t^2-4)$ *on the interval* $[0,4]$ *then we find:* $\Gamma = 66$, $\gamma = 2\exp(-4)$, $S=2$, $b-a=4$. *If we substitute these values in (13) and (21) we shall see that (21) is better than (13).*

The next Corollary gives upper and lower error bounds for the simple 3-point quadrature rule considered above.

Corollary 7 *Under the assumptions of Theorem 3 we have*

$$\frac{\gamma}{48}(b-a)^3 \le \frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a) - \int_a^b f(t)dt \le \frac{\Gamma}{48}(b-a)^3. \tag{27}$$

The above inequalities are sharp if γ, Γ *are given by (5) (*$\gamma=\gamma_2$, $\Gamma=\Gamma_2$).

Proof. The proof of (27) follows immediately from (13). It is not difficult to show that both above inequalities become equalities if we substitute $f(t)=(t-a)^2$ in (27). Thus they are sharp. ■

We now consider a perturbation of the well-known Simpson's quadrature rule. For that purpose, we define two finite sequences of harmonic (Appell-like) polynomials:

$$\begin{array}{cc} P_0(t)=1 & Q_0(t)=1 \\ P_1(t)=t-\frac{3a+\alpha(x)}{4} & Q_1(t)=t-\frac{3b+\beta(x)}{4} \\ P_2(t)=\frac{1}{2}(t-a)(t-\frac{a+\alpha(x)}{2}) & Q_2(t)=\frac{1}{2}(t-b)(t-\frac{b+\beta(x)}{2}) \\ P_3(t)=\frac{1}{6}(t-a)^2(t-\frac{a+3\alpha(x)}{4}) & Q_3(t)=\frac{1}{6}(t-b)^2(t-\frac{b+3\beta(x)}{4}) \\ P_4(t)=\frac{1}{24}(t-a)^3(t-\alpha(x)) & Q_4(t)=\frac{1}{24}(t-b)^3(t-\beta(x)) \end{array}$$

where $x\in[a,b]$ and $\alpha(x)$, $\beta(x)$ depend on x. We also define the functions

$$S_k(t)=\begin{cases} P_k(t), & a\le t\le x \\ Q_k(t), & x<t\le b, \end{cases} \tag{28}$$

for $k=0,1,2,3,4$. Additionally, we define

$$I_1(f;a,b,\alpha,\beta,x)=-\frac{1}{24}f'''(x)\left[(x-a)^3(x-\alpha(x))-(x-b)^3(x-\beta(x))\right], \tag{29}$$

$$I_2(f;a,b,\alpha,\beta,x)=\frac{1}{6}f''(x)\left[(x-a)^2(x-\frac{a+3\alpha(x)}{4})-(x-b)^2(x-\frac{b+3\beta(x)}{4})\right], \tag{30}$$

$$I_3(f;a,b,\alpha,\beta,x)=-\frac{1}{2}f'(x)\left[(x-a)(x-\frac{a+\alpha(x)}{2})-(x-b)(x-\frac{b+\beta(x)}{2})\right] \tag{31}$$

$$I_4(f;a,b,\alpha,\beta,x)=-\frac{a-\alpha(x)}{4}f(a)+\frac{3b+\beta(x)-3a-\alpha(x)}{4}f(x)-\frac{\beta(x)-b}{4}f(b), \tag{32}$$

$$I_5(f;a,b,\alpha,\beta,x)=\frac{(x-a)^4}{480}(a+4x-5\alpha(x))+\frac{(b-x)^4}{480}(b+4x-5\beta(x)). \tag{33}$$

Lemma 8 *Let S_k, $k=0,1,2,3,4$ and I_j, $j=1,2,3,4,5$ be defined by (28)–(33). If f''' : $[a,b]\to R$ is an absolutely continuous function then we have*

$$\int_a^b f(t)dt=I_1+I_2+I_3+I_4+CI_5+R(f) \tag{34}$$

where C is a constant, $I_j=I_j(f;a,b,\alpha,\beta,x)$, $j=1,...,5$ and

$$R(f)=\int_a^b\left[f^{(4)}(t)-C\right]S_4(t)dt, \tag{35}$$

$$|R(f)|\le\sup_{t\in[a,b]}\left|f^{(4)}(t)-C\right|\int_a^b|S_4(t)|\,dt. \tag{36}$$

Proof. Integrating by parts, we have

$$\begin{aligned}\int_a^b S_4(t)f''(t)dt &= -I_1-\int_a^b S_3(t)f'''(t)dt=-I_1-I_2+\int_a^b S_2f''(t)dt \\ &= -I_1-I_2-I_3-\int_a^b S_1f'(t)dt=-I_1-I_2-I_3-I_4+\int_a^b f(t)dt.\end{aligned}$$

We also have

$$\int_a^b S_4(t)dt=I_5.$$

From the above relations we see that (34)-(35) hold. The estimation (36) is obvious. ■

Remark 9 *The results of Lemma 8 can be generalized in a way given in [2] or [3]. All other observations from Remark 2 are valid in this case, too.*

Theorem 10 *Under the assumptions of Lemma 8 suppose that* $\gamma \leq f^{(4)}(t) \leq \Gamma$, $t \in [a,b]$, *where* γ, Γ *are real numbers. Then we have*

$$\left| \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right] - \int_a^b f(t)dt - \frac{\Gamma+\gamma}{5760}(b-a)^5 \right| \leq \frac{\Gamma-\gamma}{5760}(b-a)^5. \tag{37}$$

The inequality (37) is sharp.

Proof. If we choose $\alpha(x) = \frac{a+2b}{3}$, $\beta(x) = \frac{2a+b}{3}$, $x = \frac{a+b}{2}$ then we have

$$I_4(f;a,b,\frac{a+2b}{3},\frac{2a+b}{3},\frac{a+b}{2}) = \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a), \tag{38}$$

$$I_3(f;a,b,\frac{a+2b}{3},\frac{2a+b}{3},\frac{a+b}{2}) = 0, \tag{39}$$

$$I_2(f;a,b,\frac{a+2b}{3},\frac{2a+b}{3},\frac{a+b}{2}) = 0, \tag{40}$$

$$I_1(f;a,b,\frac{a+2b}{3},\frac{2a+b}{3},\frac{a+b}{2}) = 0, \tag{41}$$

$$I_5(f;a,b,\frac{a+2b}{3},\frac{2a+b}{3},\frac{a+b}{2}) = -\frac{(b-a)^5}{2880} \tag{42}$$

and

$$\int_a^b |S_4(t)|\,dt = \frac{(b-a)^5}{2880}. \tag{43}$$

From (38)–(42) and (34) with $C = \frac{\Gamma+\gamma}{2}$ it follows

$$\begin{aligned} &\int_a^b f(t)dt - \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a) + \frac{\Gamma+\gamma}{5760}(b-a)^5 \qquad (44)\\ = &\int_a^b \left[f^{(4)}(t) - \frac{\Gamma+\gamma}{2}\right] S_4(t)dt. \end{aligned}$$

From (36), (43), (44) and $\sup_{t\in[a,b]} \left|f^{(4)}(t) - \frac{\Gamma+\gamma}{2}\right| \leq \frac{\Gamma-\gamma}{2}$ we get (37).

We now show that (37) is sharp. Let us define the function

$$f(t) = \left| t - \frac{a+b}{2} \right|^r, \ r > 4.$$

Then we have

$$f^{IV}(t) = r(r-1)(r-2)(r-3) \left| t - \frac{a+b}{2} \right|^{r-4} sgn(t - \frac{a+b}{2}),$$

$$\gamma = -r(r-1)(r-2)(r-3)\frac{(b-a)^{r-4}}{2^{r-4}},$$

$$\Gamma = r(r-1)(r-2)(r-3)\frac{(b-a)^{r-4}}{2^{r-4}}$$

and

$$\int_a^b f(t)dt = \frac{(b-a)^{r+1}}{(r+1)2^r}.$$

The left-hand side of (37) becomes:

$$L.H.S.(37) = \frac{(b-a)^{r+1}}{2^r} \left| \frac{1}{r+1} - \frac{1}{3} \right|.$$

The right-hand side of (37) becomes:

$$R.H.S.(37) = \frac{1}{2^r} \frac{r(r-1)(r-2)(r-3)}{180}(b-a)^{r+1}.$$

We have

$$\lim_{r\to 4} L.H.S.(37) = \frac{(b-a)^5}{120}$$

and

$$\lim_{r\to 4} R.H.S.(37) = \frac{(b-a)^5}{120}.$$

Hence,

$$\lim_{r\to 4} L.H.S.(37) = \lim_{r\to 4} R.H.S.(37).$$

Thus, (37) is sharp. ■

Remark 11 *In the above theorem a perturbation of the Simpson's quadrature rule*

$$\int_a^b f(t)dt = \frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6}(b-a) + R(f) \tag{45}$$

is considered. Since $|R(f)| \leq \frac{\|f^{(4)}\|_\infty}{2880}(b-a)^5$ *it is not difficult to see that the perturbed rule*

$$\int_a^b f(t)dt = \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a) - \frac{\Gamma+\gamma}{5760}(b-a)^5 + R_1(f) \qquad (46)$$

has a better estimation of error. Furthermore, the rule (45) is exact for polynomials of degree ≤ 3*, while the perturbed rule (46) is exact for polynomials of degree* ≤ 5*, if we choose* γ, Γ *as in (5) (*$k=4$*,* $\gamma=\gamma_4$*,* $\Gamma=\Gamma_4$*).*

Corollary 12 *Under the assumptions of Theorem 10 we have*

$$\left| \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right] + \frac{\gamma}{2880}(b-a)^5 \right| \leq \frac{S-\gamma}{1152}(b-a)^5 \qquad (47)$$

and

$$\left| \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right] + \frac{\Gamma}{2880}(b-a)^5 \right| \leq \frac{\Gamma-S}{1152}(b-a)^5, \qquad (48)$$

where $S = \frac{f'''(b)-f'''(a)}{b-a}$.

Proof. We choose $\alpha(x)=\beta(x)=x=\frac{a+b}{2}$. Then we have

$$\int_a^b S_4(t)\left[f^{(4)}(t)-\gamma\right]dt \qquad (49)$$

$$= \int_a^b f(t)dt - \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right] + \frac{\gamma}{2880}(b-a)^5,$$

$$\left| \int_a^b S_4(t)\left[f^{(4)}(t)-\gamma\right]dt \right| \leq \max_{t\in[a,b]}|S_4(t)| \int_a^b \left|f^{(4)}(t)-\gamma\right|dt, \qquad (50)$$

$$\max_{t\in[a,b]}|S_4(t)| = \frac{(b-a)^4}{1152} \qquad (51)$$

and

$$\begin{aligned} \int_a^b \left|f^{(4)}(t)-\gamma\right|dt &= f'(b)-f'(a)-\gamma(b-a) \qquad (52)\\ &= (S-\gamma)(b-a). \end{aligned}$$

From (49)-(52) we easily get (47).

In a similar way we can prove that (48) holds. ■

Remark 13 *The above obtained estimates can be better than the estimate obtained in Theorem 10.*

The next Corollary gives upper and lower error bounds for Simpson's quadrature rule.

Corollary 14 *Under the assumptions of Theorem 4 we have*

$$\frac{\gamma}{2880}(b-a)^5 \leq \frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt \leq \frac{\Gamma}{2880}(b-a)^5. \tag{53}$$

The above inequalities are sharp if γ, Γ are given by (5) ($\gamma=\gamma_4$, $\Gamma=\Gamma_4$).

Proof. The proof of (53) follows immediately from (37). It is not difficult to show that both above inequalities become equalities if we substitute $f(t)=(t-a)^4$ in (53). Thus they are sharp. ■

3 Applications in numerical integration

Here we denote a given partition of the interval $[a,b]$ by

$$\pi=\{x_0=a<x_1<\cdots<x_n=b\}.$$

Theorem 15 *Let the assumptions of Theorem 3 hold. If π is a given partition of the interval $[a,b]$ and $h_i=x_{i+1}-x_i$, then we have*

$$\int_a^b f(t)dt = A_M(f,\pi)+R_M(f,\pi), \tag{54}$$

where

$$A_M(f,\pi)=\frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+2f(\frac{x_i+x_{i+1}}{2})+f(x_{i+1})\right]h_i-\frac{\Gamma+\gamma}{96}\sum_{i=0}^{n-1}h_i^3 \tag{55}$$

and

$$|R_M(f,\pi)| \leq \frac{\Gamma-\gamma}{96}\sum_{i=0}^{n-1}h_i^3. \tag{56}$$

Proof. From (18), with $a=x_i$ and $b=x_{i+1}$ we get

$$\begin{aligned}&\int_{x_i}^{x_{i+1}} p_i(t)\left[f''(t)-\frac{\Gamma+\gamma}{2}\right]dt \qquad (57)\\ =\ &\int_{x_i}^{x_{i+1}} f(t)dt-\frac{f(x_i)+2f(\frac{x_i+x_{i+1}}{2})+f(x_{i+1})}{4}h_i+\frac{\Gamma+\gamma}{96}h_i^3,\end{aligned}$$

where

$$p_i(t) = \begin{cases} \frac{t-x_i}{2}(t - \frac{x_i+x_{i+1}}{2}), & t \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right] \\ \frac{t-x_{i+1}}{2}(t - \frac{x_i+x_{i+1}}{2}), & t \in \left(\frac{x_i+x_{i+1}}{2}, x_{i+1}\right] \end{cases},$$

for $i = 0, 1, ..., n-1$ ($p_i(t)$ corresponds to $S_2(t)$).

From (13) we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 2f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{4} h_i + \frac{\Gamma+\gamma}{96} h_i^3 \right| \leq \frac{\Gamma-\gamma}{96} h_i^3, \quad (58)$$

for $i = 0, 1, ..., n-1$.

If we now sum (57) over i from 0 to $n-1$ and apply the triangle inequality and (58) then we get (54)-(56). ■

Theorem 16 *Let the assumptions of Theorem 3 hold. If π is a given partition of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$, $S_i = \frac{f'(x_{i+1})-f'(x_i)}{h_i}$, $i = 0, 1, ..., n-1$, then we have*

$$\int_a^b f(t)dt = A_N(f, \pi) + R_N(f, \pi), \quad (59)$$

where

$$A_N(f, \pi) = \frac{1}{4} \sum_{i=0}^{n-1} \left[f(x_i) + 2f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right] h_i - \frac{\gamma}{48} \sum_{i=0}^{n-1} h_i^3 \quad (60)$$

and

$$|R_N(f, \pi)| \leq \frac{1}{32} \sum_{i=0}^{n-1} (S_i - \gamma) h_i^3. \quad (61)$$

We also have

$$\int_a^b f(t)dt = A_P(f, \pi) + R_P(f, \pi), \quad (62)$$

where

$$A_P(f, \pi) = \frac{1}{4} \sum_{i=0}^{n-1} \left[f(x_i) + 2f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right] h_i - \frac{\Gamma}{48} \sum_{i=0}^{n-1} h_i^3 \quad (63)$$

and

$$|R_P(f, \pi)| \leq \frac{1}{32} \sum_{i=0}^{n-1} (\Gamma - S_i) h_i^3. \quad (64)$$

Proof. From (23), with $a = x_i$ and $b = x_{i+1}$ we get

$$\int_{x_i}^{x_{i+1}} p_i(t)\left[f''(t) - \gamma\right] dt \tag{65}$$
$$= \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 2f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{4} h_i + \frac{\gamma}{48} h_i^3,$$

for $i = 0, 1, ..., n-1$ ($p_i(t)$ are defined in the proof of Theorem 15).

From (21) we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 2f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{4} h_i + \frac{\gamma}{48} h_i^3 \right| \leq \frac{S_i - \gamma}{32} h_i^3, \tag{66}$$

for $i = 0, 1, ..., n-1$.

If we now sum (65) over i from 0 to $n-1$ and apply the triangle inequality and (66) then we get (59)-(61).

In a similar way we get (62)-(64). ■

Theorem 17 *Let the assumptions of Theorem 10 hold. If π is a given partition of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$, then we have*

$$\int_a^b f(t)dt = A_S(f, \pi) + R_S(f, \pi), \tag{67}$$

where

$$A_S(f, \pi) = \frac{1}{6} \sum_{i=0}^{n-1} \left[f(x_i) + 4f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right] h_i - \frac{\Gamma + \gamma}{5760} \sum_{i=0}^{n-1} h_i^5 \tag{68}$$

and

$$|R_S(f, \pi)| \leq \frac{\Gamma - \gamma}{5760} \sum_{i=0}^{n-1} h_i^5. \tag{69}$$

Proof. From (44), with $a = x_i$ and $b = x_{i+1}$ we get

$$\int_{x_i}^{x_{i+1}} p_i(t) \left[f^{(4)}(t) - \frac{\Gamma + \gamma}{2} \right] dt \tag{70}$$
$$= \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 4f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{6} h_i + \frac{\Gamma + \gamma}{5760} h_i^5,$$

where

$$p_i(t) = \begin{cases} \frac{(t-x_i)^3}{24}(t - \frac{x_i+2x_{i+1}}{3}), & t \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right] \\ \frac{(t-x_{i+1})^3}{24}(t - \frac{2x_i+x_{i+1}}{3}), & t \in \left(\frac{x_i+x_{i+1}}{2}, x_{i+1}\right] \end{cases},$$

for $i = 0, 1, ..., n-1$ ($p_i(t)$ corresponds to $S_4(t)$).

From (37) we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 4f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{6} h_i + \frac{\Gamma+\gamma}{5760} h_i^5 \right| \leq \frac{\Gamma-\gamma}{5760} h_i^5, \quad (71)$$

for $i = 0, 1, ..., n-1$.

If we now sum (70) over i from 0 to $n-1$ and apply the triangle inequality and (71) then we get (67)-(69). ■

Theorem 18 *Let the assumptions of Theorem 10 hold. If π is a given partition of the interval $[a, b]$ and $h_i = x_{i+1} - x_i$, $S_i = \frac{f'''(x_{i+1}) - f'''(x_i)}{h_i}$, $i = 0, 1, ..., n-1$, then we have*

$$\int_a^b f(t)dt = A_T(f, \pi) + R_T(f, \pi), \quad (72)$$

where

$$A_T(f, \pi) = \frac{1}{6} \sum_{i=0}^{n-1} \left[f(x_i) + 4f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right] h_i - \frac{\gamma}{2880} \sum_{i=0}^{n-1} h_i^5 \quad (73)$$

and

$$|R_T(f, \pi)| \leq \frac{1}{1152} \sum_{i=0}^{n-1} (S_i - \gamma) h_i^5. \quad (74)$$

We also have

$$\int_a^b f(t)dt = A_U(f, \pi) + R_U(f, \pi), \quad (75)$$

where

$$A_U(f, \pi) = \frac{1}{6} \sum_{i=0}^{n-1} \left[f(x_i) + 4f(\frac{x_i + x_{i+1}}{2}) + f(x_{i+1}) \right] h_i - \frac{\Gamma}{2880} \sum_{i=0}^{n-1} h_i^5 \quad (76)$$

and

$$|R_U(f, \pi)| \leq \frac{1}{1152} \sum_{i=0}^{n-1} (\Gamma - S_i) h_i^5. \quad (77)$$

Proof. From (49), with $a = x_i$ and $b = x_{i+1}$ we get

$$\int_{x_i}^{x_{i+1}} p_i(t) \left[f^{(4)}(t) - \gamma \right] dt \tag{78}$$
$$= \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 4f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{6} h_i + \frac{\gamma}{2880} h_i^5,$$

for $i = 0, 1, ..., n-1$ ($p_i(t)$ are defined in the proof of Theorem 17).

From (47) we have

$$\left| \int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i) + 4f(\frac{x_i+x_{i+1}}{2}) + f(x_{i+1})}{6} h_i + \frac{\gamma}{2880} h_i^5 \right| \leq \frac{S_i - \gamma}{1152} h_i^5, \tag{79}$$

for $i = 0, 1, ..., n-1$.

If we now sum (78) over i from 0 to $n-1$ and apply the triangle inequality and (79) then we get (72)-(74).

In a similar way we get (75)-(77) ■

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Construction of Upper and Lower Solutions with Applications to Singular Boundary Value Problems

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Abstract

An upper and lower solution theory is presented for the Dirichlet boundary value problem $y'' + f(t, y, y') = 0,\ 0 < t < 1$ with $y(0) = y(1) = 0$. Our nonlinearity may be singular in its dependent variable and is allowed to change sign.

Keywords: Boundary value problem, upper and lower solutions, singular, existence.

Subject Classes: 34B16.

1. Introduction

An approach based on upper and lower solutions and a truncation technique is presented for the singular boundary value problem

$$\begin{cases} y'' + q(t)\, f(t, y, y') = 0, \quad 0 < t < 1 \\ y(0) = 0 = y(1), \end{cases} \tag{1.1}$$

where our nonlinearity f is allowed to change sign. In addition f may not be a Carathéodory function because of the singular behavior of the y variable i.e. f may be singular at $y = 0$. In the literature the case when f is independent

of its third variable (i.e. when $f(t,y,z) \equiv f(t,y)$) has received almost all the attention, see [2-4, 6, 7] and the references therein. Only a few papers [1, 8] have appeared when f depends on the y' variable. This paper presents a new and very general result for (1.1) when $f : (0,1)\times(0,\infty)\times\mathbf{R} \to \mathbf{R}$. In addition our results are new even when f is independent of the third variable. It is also worth remarking here that we could consider Sturm-Liouville boundary data in (1.1); however since the arguments are essentially the same we will restrict our discussion to Dirichlet boundary data.

2. Existence Theory

In this section we present an upper and lower solution theory for the Dirichlet singular boundary value problem

$$\begin{cases} y'' + q(t)\, f(t,y,y') = 0, \quad 0 < t < 1 \\ y(0) = y(1) = 0, \end{cases} \tag{2.1}$$

where our nonlinearity f may change sign.

Theorem 2.1. *Let $n_0 \in \{1,2,...\}$ be fixed and suppose the following conditions are satisfied:*

$$f : (0,1)\times(0,\infty)\times\mathbf{R} \to \mathbf{R} \quad \textit{is continuous} \tag{2.2}$$

$$q \in C(0,1)\cap L^1[0,1] \quad \textit{with} \quad q > 0 \quad \textit{on} \quad (0,1) \tag{2.3}$$

$$\begin{cases} \textit{let} \quad n \in \{n_0, n_0+1, ...\} \quad \textit{and associated with each} \quad n \quad \textit{we} \\ \textit{have a constant} \quad \rho_n \quad \textit{such that} \quad \{\rho_n\} \quad \textit{is a nonincreasing} \\ \textit{sequence with} \quad \lim_{n\to\infty} \rho_n = 0 \quad \textit{and such that for} \\ \frac{1}{2^{n+1}} \le t \le 1 \quad \textit{and} \quad z \in \mathbf{R} \quad \textit{we have} \quad f(t,\rho_n,z) \ge 0 \end{cases} \tag{2.4}$$

$$\begin{cases} \exists\ \alpha \in C[0,1]\cap C^2(0,1) \quad \textit{with} \quad \alpha(0) = \alpha(1) = 0, \\ \alpha > 0 \quad \textit{on} \quad (0,1) \quad \textit{such that} \\ q(t)\, f(t,\alpha(t),z) + \alpha''(t) \ge 0 \quad \textit{for} \quad (t,z) \in (0,1)\times\mathbf{R} \end{cases} \tag{2.5}$$

$$\begin{cases} \exists\ \beta \in C^1[0,1]\cap C^2(0,1) \quad \textit{with} \quad \beta(t) \ge \alpha(t),\ \beta(t) \ge \rho_{n_0} \\ \textit{for} \quad t \in [0,1] \quad \textit{with} \quad q(t)\, f(t,\beta(t),\beta'(t)) + \beta''(t) \le 0 \\ \textit{for} \quad t \in (0,1) \quad \textit{and} \quad q(t)\, f\left(\frac{1}{2^{n_0+1}},\beta(t),\beta'(t)\right) + \beta''(t) \le 0 \\ \textit{for} \quad t \in \left(0, \frac{1}{2^{n_0+1}}\right) \end{cases} \tag{2.6}$$

$$(2.7)\qquad \begin{cases} \textit{for any } \epsilon > 0,\ \epsilon < a_0 = \sup_{t\in[0,1]} \beta(t),\ \exists \textit{ a function} \\ \psi_\epsilon \textit{ continuous on } [0,\infty) \textit{ with } |f(t,y,z)| \le \psi_\epsilon(|z|) \\ \textit{for } (t,y,z) \in (0,1)\times[\epsilon, a_0]\times \mathbf{R} \end{cases}$$

and

$$(2.8)\qquad \textit{for any } \epsilon > 0,\ \epsilon < a_0, \quad \textit{we have } \int_0^1 q(s)\,ds < \int_0^\infty \frac{du}{\psi_\epsilon(u)}.$$

Then (2.1) *has a solution* $y \in C[0,1]\cap C^2(0,1)$ *with* $\alpha(t) \le y(t) \le \beta(t)$ *for* $t \in [0,1]$.

PROOF: For $n = n_0,\ n_0+1, ...$ let

$$e_n = \left[\frac{1}{2^{n+1}}, 1\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}\,,\ t\right\},\ 0 \le t \le 1$$

and

$$f_n(t,x,z) = \max\left\{f(\theta_n(t),x,z)\,,\ f(t,x,z)\right\}.$$

Next we define inductively

$$g_{n_0}(t,x,z) = f_{n_0}(t,x,z)$$

and

$$g_n(t,x,z) = \min\left\{f_{n_0}(t,x,z), ..., f_n(t,x,z)\right\},\ n = n_0+1,\ n_0+2, ...\ .$$

Notice

$$f(t,x,z) \le ... \le g_{n+1}(t,x,z) \le g_n(t,x,z) \le ... \le g_{n_0}(t,x,z)$$

for $(t,x,z) \in (0,1)\times(0,\infty)\times\mathbf{R}$ and

$$g_n(t,x,z) = f(t,x,z) \quad \text{for} \quad (t,x,z) \in e_n \times (0,\infty)\times\mathbf{R}.$$

Without loss of generality assume $\rho_{n_0} \le \min_{t\in\left[\frac{1}{3},\frac{2}{3}\right]} \alpha(t)$. Fix $n \in \{n_0, n_0+1, ...\}$. Let $t_n \in \left[0,\frac{1}{3}\right]$ and $s_n \in \left[\frac{2}{3},1\right]$ be such that

$$\alpha(t_n) = \alpha(s_n) = \rho_n \quad \text{and} \quad \alpha(t) \le \rho_n \quad \text{for} \quad t \in [0,t_n]\cup[s_n,1].$$

Define

$$\alpha_n(t) = \begin{cases} \rho_n & \text{if } t \in [0,t_n]\cup[s_n,1] \\ \alpha(t) & \text{if } t \in (t_n,s_n). \end{cases}$$

We begin with the boundary value problem

$$\begin{cases} y'' + q(t)\, g^{\star}_{n_0}(t, y, y') = 0,\ 0 < t < 1 \\ y(0) = y(1) = \rho_{n_0}; \end{cases} \tag{2.9}$$

here

$$g^{\star}_{n_0}(t, y, z) = \begin{cases} g_{n_0}(t, \alpha_{n_0}(t), z^{\star}) + r(\alpha_{n_0}(t) - y), & y \le \alpha_{n_0}(t) \\ g_{n_0}(t, y, z^{\star}), & \alpha_{n_0}(t) \le y \le \beta(t) \\ g_{n_0}(t, \beta(t), z^{\star}) + r(\beta(t) - y), & y \ge \beta(t), \end{cases}$$

with

$$z^{\star} = \begin{cases} M_{n_0}, & z > M_{n_0} \\ z, & -M_{n_0} \le z \le M_{n_0} \\ -M_{n_0}, & z < -M_{n_0} \end{cases}$$

and $r : \mathbf{R} \to [-1, 1]$ the radial retraction defined by

$$r(u) = \begin{cases} u, & |u| \le 1 \\ \frac{u}{|u|}, & |u| > 1, \end{cases}$$

and $M_{n_0} \ge \sup_{[0,1]} |\beta'(t)|$ is a predetermined constant (see (2.15)). Now Schauder's fixed point theorem [7] guarantees that there exists a solution $y_{n_0} \in C^1[0, 1]$ to (2.9). We first show

$$y_{n_0}(t) \ge \alpha_{n_0}(t),\ \ t \in [0, 1]. \tag{2.10}$$

Suppose (2.10) is not true. Then $y_{n_0} - \alpha_{n_0}$ has a negative absolute minimum at $\tau \in (0, 1)$. Now since $y_{n_0}(0) - \alpha_{n_0}(0) = 0 = y_{n_0}(1) - \alpha_{n_0}(1)$ there exists $\tau_0,\ \tau_1 \in [0, 1]$ with $\tau \in (\tau_1, \tau_2)$ and

$$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = y_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0$$

and

$$y_{n_0}(t) - \alpha_{n_0}(t) < 0,\ \ t \in (\tau_0, \tau_1).$$

We now <u>claim</u>

$$(y_{n_0} - \alpha_{n_0})''(t) \ < \ 0 \ \text{ for a.e. } \ t \in (\tau_0, \tau_1). \tag{2.11}$$

If (2.11) is true then

$$y_{n_0}(t) - \alpha_{n_0}(t) = -\int_{\tau_0}^{\tau_1} G(t, s)\, [y''_{n_0}(s) - \alpha''_{n_0}(s)]\, ds \quad \text{for} \ \ t \in (\tau_0, \tau_1)$$

with

$$G(t,s) = \begin{cases} \frac{(s-\tau_0)(\tau_1-t)}{\tau_1-\tau_0}, & \tau_0 \le s \le t \\ \frac{(t-\tau_0)(\tau_1-s)}{\tau_1-\tau_0}, & t \le s \le \tau_1 \end{cases}$$

so we have

$$y_{n_0}(t) - \alpha_{n_0}(t) > 0 \quad \text{for} \quad t \in (\tau_0, \tau_1),$$

a contradiction. As a result if we show that (2.11) is true then (2.10) will follow. To see (2.11) we will show

$$(y_{n_0} - \alpha_{n_0})''(t) < 0 \quad \text{for} \quad t \in (\tau_0, \tau_1) \quad \text{provided} \quad t \ne t_{n_0} \quad \text{or} \quad t \ne s_{n_0}.$$

Fix $t \in (\tau_0, \tau_1)$ and assume $t \ne t_{n_0}$ or $t \ne s_{n_0}$. Then

$$\begin{aligned} (y_{n_0} - \alpha_{n_0})''(t) &= -[q(t)\{ g_{n_0}(t, \alpha_{n_0}(t), (y'_{n_0}(t))^\star) \\ &\qquad + r(\alpha_{n_0}(t) - y_{n_0}(t))\} + \alpha''_{n_0}(t)] \\ &= \begin{cases} -[q(t)\left\{g_{n_0}(t, \alpha(t), (y'_{n_0}(t))^\star) + r(\alpha(t) - y_{n_0}(t))\right\} \\ \quad +\alpha''(t)] & \text{if} \quad t \in (t_{n_0}, s_{n_0}) \\ -\left[q(t)\left\{g_{n_0}(t, \rho_{n_0}, (y'_{n_0}(t))^\star) + r(\rho_{n_0} - y_{n_0}(t))\right\}\right] \\ & \text{if} \quad t \in (0, t_{n_0}) \cup (s_{n_0}, 1). \end{cases} \end{aligned}$$

Case (A). $t \in \left[\frac{1}{2^{n_0+1}}, 1\right)$.

Then since $g_{n_0}(t,x,z) = f(t,x,z)$ for $(x,z) \in (0,\infty) \times \mathbf{R}$ (note $t \in e_{n_0}$) we have

$$\begin{aligned} (y_{n_0} - \alpha_{n_0})''(t) &= \begin{cases} -[q(t)\left\{f(t, \alpha(t), (y'_{n_0}(t))^\star) + r(\alpha(t) - y_{n_0}(t))\right\} \\ \quad +\alpha''(t)] & \text{if} \quad t \in (t_{n_0}, s_{n_0}) \\ -\left[q(t)\left\{f(t, \rho_{n_0}, (y'_{n_0}(t))^\star) + r(\rho_{n_0} - y_{n_0}(t))\right\}\right] \\ & \text{if} \quad t \in (0, t_{n_0}) \cup (s_{n_0}, 1) \end{cases} \\ &< 0, \end{aligned}$$

from (2.4) and (2.5).

Case (B). $t \in \left(0, \frac{1}{2^{n_0+1}}\right)$.

Then since

$$g_{n_0}(t,x,z) = \max\left\{ f\left(\frac{1}{2^{n_0+1}}, x, z\right), f(t,x,z)\right\}$$

we have

$$g_{n_0}(t,x,z) \ge f(t,x) \quad \text{and} \quad g_{n_0}(t,x,z) \ge f\left(\frac{1}{2^{n_0+1}}, x, z\right)$$

for $(x,z) \in (0,\infty) \times \mathbf{R}$. Thus we have

$$\begin{aligned}(y_{n_0}-\alpha_{n_0})''(t) &\leq \begin{cases} -[q(t)\left\{f(t,\alpha(t),(y'_{n_0}(t))^\star)+r(\alpha(t)-y_{n_0}(t))\right\} \\ \quad +\alpha''(t)] & \text{if } \ t\in(t_{n_0},s_{n_0}) \\ -[q(t)\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0},(y'_{n_0}(t))^\star\right) \\ \quad +r(\rho_{n_0}-y_{n_0}(t))\}] & \text{if } \ t\in(0,t_{n_0})\cup(s_{n_0},1) \end{cases} \\ &< 0,\end{aligned}$$

from (2.4) and (2.5).

Consequently (2.11) (and so (2.10)) holds and now since $\alpha(t) \leq \alpha_{n_0}(t)$ for $t \in [0,1]$ we have

$$\alpha(t) \leq \alpha_{n_0}(t) \leq y_{n_0}(t) \quad \text{for} \quad t \in [0,1]. \tag{2.12}$$

Next we show

$$y_{n_0}(t) \leq \beta(t) \quad \text{for} \quad t \in [0,1]. \tag{2.13}$$

If (2.13) is not true then $y_{n_0}-\beta$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$, in which case $(y_{n_0}-\beta)'(\tau_0)=0$ and $(y_{n_0}-\beta)''(\tau_0) \leq 0$. There are two cases to consider, namely $\tau_0 \in \left[\frac{1}{2^{n_0+1}},1\right)$ and $\tau_0 \in \left(0,\frac{1}{2^{n_0+1}}\right)$.

Case (A). $\tau_0 \in \left[\frac{1}{2^{n_0+1}},1\right)$.

Then $y_{n_0}(\tau_0) > \beta(\tau_0)$, $y'_{n_0}(\tau_0)=\beta'(\tau_0)$ together with $g_{n_0}(\tau_0,x,z) = f(\tau_0,x,z)$ for $(x,z) \in (0,\infty)\times\mathbf{R}$ and $M_{n_0} \geq \sup_{[0,1]}|\beta'(t)|$ gives

$$\begin{aligned}(y_{n_0}-\beta)''(\tau_0) &= -q(\tau_0)\left[g_{n_0}(\tau_0,\beta(\tau_0),(y'_{n_0}(\tau_0))^\star)+r(\beta(\tau_0)-y_{n_0}(\tau_0))\right] \\ &- \beta''(\tau_0) \\ &= -q(\tau_0)\left[f(\tau_0,\beta(\tau_0),\beta'(\tau_0))+r(\beta(\tau_0)-y_{n_0}(\tau_0))\right] \\ &- \beta''(\tau_0) \\ &> 0\end{aligned}$$

from (2.6), a contradiction.

Case (B). $\tau_0 \in \left(0,\frac{1}{2^{n_0+1}}\right)$.

Now

$$g_{n_0}(\tau_0,x,z) = \max\left\{f\left(\frac{1}{2^{n_0+1}},x,z\right),\ f(\tau_0,x,z)\right\}$$

for $(x,z) \in (0,\infty)\times\mathbf{R}$ gives

$$\begin{aligned}(y_{n_0}-\beta)''(\tau_0) &= -q(\tau_0)\left[\max\{f(\frac{1}{2^{n_0+1}},\beta(\tau_0),\beta'(\tau_0)),f(\tau_0,\beta(\tau_0),\beta'(\tau_0))\}\right. \\ &+ \left. r(\beta(\tau_0)-y_{n_0}(\tau_0))\right]-\beta''(\tau_0) \\ &> 0\end{aligned}$$

from (2.6), a contradiction.

Thus (2.13) holds. Next we show

$$|y'_{n_0}|_\infty = \sup_{[0,1]} |y'_{n_0}(t)| \le M_{n_0}. \tag{2.14}$$

With $\epsilon = \min_{[0,1]} \alpha_{n_0}(t)$, then (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \le \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times [\epsilon, a_0] \times \mathbf{R}$$

where $a_0 = \sup_{[0,1]} \beta(t)$. Let $M_{n_0} \ge \sup_{[0,1]} |\beta'(t)|$ be chosen so that

$$\int_0^1 q(s)\, ds \; < \; \int_0^{M_{n_0}} \frac{du}{\psi_\epsilon(u)} \tag{2.15}$$

holds. Suppose (2.14) is false. Without loss of generality assume $y'_{n_0}(t) \not\le M_{n_0}$ for some $t \in [0,1]$. Then since $y_{n_0}(0) = y_{n_0}(1) = \rho_{n_0}$ there exists $\tau_1 \in (0,1)$ with $y'_{n_0}(\tau_1) = 0$, and so there exists τ_2, $\tau_3 \in (0,1)$ with $y'_{n_0}(\tau_3) = 0$, $y'_{n_0}(\tau_2) = M_{n_0}$ and $0 \le y'_{n_0}(s) \le M_{n_0}$ for s between τ_3 and τ_2. Without loss of generality assume $\tau_3 < \tau_2$. Now since $\alpha_{n_0}(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in [0,1]$ and

$$g_{n_0}(t,x,z) = \max\left\{ f\left(\frac{1}{2^{n_0+1}}, x, z\right),\, f(t,x,z)\right\}$$

for $(t,x,z) \in (0,1) \times (0,\infty) \times \mathbf{R}$, we have for $s \in (\tau_3, \tau_2)$ that

$$y''_{n_0}(s) \le q(s)\, \psi_\epsilon(y'_{n_0}(s)),$$

and so

$$\int_0^{M_{n_0}} \frac{du}{\psi_\epsilon(u)} = \int_{\tau_3}^{\tau_2} \frac{y''_{n_0}(s)}{\psi_\epsilon(y'_{n_0}(s)}\, ds \le \int_0^1 q(s)\, ds.$$

This contradicts (2.15). The other cases are treated similarly. As a result $\alpha(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in [0,1]$ and $|y'_{n_0}|_\infty \le M_{n_0}$. Thus y_{n_0} satisfies $y''_{n_0} + q\, g_{n_0}(t, y_{n_0}, y'_{n_0}) = 0$ on $(0,1)$.

Next we consider the boundary value problem

$$\begin{cases} y'' + q(t)\, g^\star_{n_0+1}(t,y,y') = 0,\ 0 < t < 1 \\ y(0) = y(1) = \rho_{n_0+1} \end{cases} \tag{2.16}$$

where

$$g^\star_{n_0+1}(t,y,z) = \begin{cases} g_{n_0+1}(t, \alpha_{n_0+1}(t), z^\star) + r(\alpha_{n_0+1}(t) - y), & y \le \alpha_{n_0+1}(t) \\ g_{n_0+1}(t,y,z^\star), & \alpha_{n_0+1}(t) \le y \le y_{n_0}(t) \\ g_{n_0+1}(t, y_{n_0}(t), z^\star) + r(y_{n_0}(t) - y), & y \ge y_{n_0}(t) \end{cases}$$

with

$$z^\star = \begin{cases} M_{n_0+1}, & z > M_{n_0+1} \\ z, & -M_{n_0+1} \le z \le M_{n_0+1} \\ -M_{n_0+1}, & z < -M_{n_0+1}; \end{cases}$$

here $M_{n_0+1} \ge M_{n_0}$ is a predetermined constant (see (2.22)). Now Schauder's fixed point theorem guarantees that there exists a solution $y_{n_0+1} \in C^1[0,1]$ to (2.16). We first show

$$y_{n_0+1}(t) \ge \alpha_{n_0+1}(t), \quad t \in [0,1]. \tag{2.17}$$

Suppose (2.17) is not true. Then there exists $\tau_0,\ \tau_1 \in [0,1]$ with

$$y_{n_0+1}(\tau_0) - \alpha_{n_0+1}(\tau_0) = y_{n_0+1}(\tau_1) - \alpha_{n_0+1}(\tau_1) = 0$$

and

$$y_{n_0+1}(t) - \alpha_{n_0+1}(t) < 0, \quad t \in (\tau_0, \tau_1).$$

If we show

$$(y_{n_0+1} - \alpha_{n_0+1})''(t) \ < \ 0 \ \text{ for a.e. } \ t \in (\tau_0, \tau_1), \tag{2.18}$$

then as before (2.17) is true. Fix $t \in (\tau_0, \tau_1)$ and assume $t \ne t_{n_0+1}$ or $t \ne s_{n_0+1}$. Then

$$(y_{n_0+1} - \alpha_{n_0+1})''(t) \ = \ \begin{cases} -\,[q(t)\,\{g_{n_0+1}(t, \alpha(t), (y'_{n_0+1}(t))^\star) \\ \quad +r(\alpha(t) - y_{n_0+1}(t))\} + \alpha''(t)] \\ \qquad \text{if} \quad t \in (t_{n_0+1}, s_{n_0+1}) \\ -\,[\,q(t)\,\{g_{n_0+1}(t, \rho_{n_0+1}, (y'_{n_0+1}(t))^\star) \\ \quad +r(\rho_{n_0+1} - y_{n_0+1}(t))\}\,] \\ \qquad \text{if} \quad t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1). \end{cases}$$

Case (A). $t \in \left[\frac{1}{2^{n_0+2}}, 1\right)$.

Then since $g_{n_0+1}(t,x,z) = f(t,x,z)$ for $(x,z) \in (0,\infty) \times \mathbf{R}$ (note $t \in e_{n_0+1}$) we have

$$\begin{aligned} (y_{n_0+1} - \alpha_{n_0+1})''(t) \ &= \ \begin{cases} -\,[q(t)\,\{f(t, \alpha(t), (y'_{n_0+1}(t))^\star) \\ \quad +r(\alpha(t) - y_{n_0+1}(t))\} + \alpha''(t)] \\ \qquad \text{if} \quad t \in (t_{n_0+1}, s_{n_0+1}) \\ -\,[\,q(t)\,\{f(t, \rho_{n_0+1}, (y'_{n_0+1}(t))^\star) \\ \quad +r(\rho_{n_0+1} - y_{n_0+1}(t))\}\,] \\ \qquad \text{if} \quad t \in (0, t_{n_0+1}) \cup (s_{n_0+1}, 1) \end{cases} \\ &< \ 0, \end{aligned}$$

from (2.4) and (2.5).

Case (B). $t \in \left(0\,,\,\frac{1}{2^{n_0+2}}\right)$.

Then since $g_{n_0+1}(t,x,z)$ equals

$$\min\{\max\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\,,\,f(t,x,z)\}\,,\,\max\{f\left(\frac{1}{2^{n_0+2}},x,z\right)\,,\,f(t,x,z)\}\}$$

we have

$$g_{n_0+1}(t,x,z) \geq f(t,x,z)$$

and

$$g_{n_0+1}(t,x,z) \geq \min\left\{f\left(\frac{1}{2^{n_0+1}},x,z\right)\,,\,f\left(\frac{1}{2^{n_0+2}},x,z\right)\right\}$$

for $(x,z) \in (0,\infty)\times\mathbf{R}$. Thus we have

$$\begin{aligned}(y_{n_0+1}-\alpha_{n_0+1})''(t) \;&\leq\; \begin{cases} -[q(t)\{f(t,\alpha(t),(y'_{n_0+1}(t))^\star) \\ +r(\alpha(t)-y_{n_0+1}(t))\}+\alpha''(t)] \\ \qquad \text{if} \quad t\in(t_{n_0+1},s_{n_0+1}) \\ -[q(t)\{\min\{f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},(y'_{n_0+1}(t))^\star\right)\,, \\ f\left(\frac{1}{2^{n_0+2}},\rho_{n_0+1},(y'_{n_0+1}(t))^\star\right)\} \\ +r(\rho_{n_0+1}-y_{n_0+1}(t))\}] \\ \qquad \text{if} \quad t\in(0,t_{n_0+1})\cup(s_{n_0+1},1) \end{cases} \\ &<\; 0,\end{aligned}$$

from (2.4) and (2.5) since $f\left(\frac{1}{2^{n_0+1}},\rho_{n_0+1},(y'_{n_0+1}(t))^\star\right) \geq 0$ because

$$f\left(t,\rho_{n_0+1},(y'_{n_0+1}(t))^\star\right) \geq 0 \quad \text{for} \quad t\in\left[\frac{1}{2^{n_0+2}}\,,\,1\right]$$

and

$$\frac{1}{2^{n_0+1}} \in \left[\frac{1}{2^{n_0+2}}\,,\,1\right].$$

Consequently (2.17) is true so

$$\alpha(t) \leq \alpha_{n_0+1}(t) \leq y_{n_0+1}(t) \quad \text{for} \quad t\in[0,1]. \tag{2.19}$$

Next we show

$$y_{n_0+1}(t) \leq y_{n_0}(t) \quad \text{for} \quad t\in[0,1]. \tag{2.20}$$

If (2.20) is not true then $y_{n_0+1}-y_{n_0}$ would have a positive absolute maximum at say $\tau_0\in(0,1)$, in which case

$$(y_{n_0+1}-y_{n_0})'(\tau_0)=0 \quad \text{and} \quad (y_{n_0+1}-y_{n_0})''(\tau_0)\leq 0.$$

Then $y_{n_0+1}(\tau_0) > y_{n_0}(\tau_0)$ together with $g_{n_0}(\tau_0,x,z) \geq g_{n_0+1}(\tau_0,x,z)$ for $(x,z) \in (0,\infty)\times\mathbf{R}$ gives (note $(y'_{n_0+1}(\tau_0))^\star = (y'_{n_0}(\tau_0))^\star = y'_{n_0}(\tau_0)$ since $M_{n_0+1} \geq M_{n_0}$ and $|y'_{n_0}|_\infty \leq M_{n_0}$),

$$\begin{aligned}(y_{n_0+1}-y_{n_0})''(\tau_0) &= -q(\tau_0)\,[g_{n_0+1}(\tau_0,y_{n_0}(\tau_0),(y'_{n_0+1}(\tau_0))^\star)\\ &+ r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))] - y''_{n_0}(\tau_0)\\ &\geq -q(\tau_0)\,[g_{n_0}(\tau_0,y_{n_0}(\tau_0),y'_{n_0}(\tau_0))\\ &+ r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))] - y''_{n_0}(\tau_0)\\ &= -q(\tau_0)\,[r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))]\\ &> 0,\end{aligned}$$

a contradiction. Thus (2.20) holds. Next we show

$$|y'_{n_0+1}|_\infty \leq M_{n_0+1}. \tag{2.21}$$

With $\epsilon = \min_{[0,1]}\alpha_{n_0+1}(t)$, then (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \leq \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z)\in(0,1)\times[\epsilon,a_0]\times\mathbf{R}$$

where $a_0 = \sup_{[0,1]}\beta(t)$. Let $M_{n_0+1} \geq M_{n_0}$ be chosen so that

$$\int_0^1 q(s)\,ds < \int_0^{M_{n_0+1}} \frac{du}{\psi_\epsilon(u)}. \tag{2.22}$$

Essentially the same argument as before guarantees that (2.21) holds. Thus $y''_{n_0+1} + q\,g_{n_0+1}(t,y_{n_0+1},y'_{n_0+1}) = 0$ on $(0,1)$.

Now proceed inductively to construct $y_{n_0+2}, y_{n_0+3}, \ldots.$ as follows. Suppose we have y_k for some $k \in \{n_0+1, n_0+2, \ldots\}$ with $\alpha(t) \leq \alpha_k(t) \leq y_k(t) \leq y_{k-1}(t)\,(\leq \beta(t))$ for $t\in[0,1]$. Then consider the boundary value problem

$$\begin{cases} y'' + q(t)\,g^\star_{k+1}(t,y,y') = 0,\ 0<t<1\\ y(0) = y(1) = \rho_{k+1}\end{cases} \tag{2.23}$$

where

$$g^\star_{k+1}(t,y,z) = \begin{cases} g_{k+1}(t,\alpha_{k+1}(t),z^\star) + r(\alpha_{k+1}(t)-y), & y \leq \alpha_{k+1}(t)\\ g_{k+1}(t,y,z^\star), & \alpha_{k+1}(t)\leq y\leq y_k(t)\\ g_{k+1}(t,y_k(t),z^\star) + r(y_k(t)-y), & y\geq y_k(t)\end{cases}$$

with

$$z^\star = \begin{cases} M_{k+1}, & z > M_{k+1}\\ z, & -M_{k+1}\leq z\leq M_{k+1}\\ -M_{k+1}, & z<-M_{k+1};\end{cases}$$

here $M_{k+1} \geq M_k$ is a predetermined constant. Now Schauder's fixed point theorem guarantees that (2.23) has a solution $y_{k+1} \in C^1[0,1]$, and essentially the same reasoning as above yields

$$\alpha(t) \leq \alpha_{k+1}(t) \leq y_{k+1}(t) \leq y_k(t) \quad \text{for} \quad t \in [0,1], \quad |y'_{k+1}|_\infty \leq M_{k+1},$$

so $y''_{k+1} + q\, g_{k+1}(t, y_{k+1}, y'_{k+1}) = 0$ on $(0,1)$.

Now lets look at the interval $[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}]$. We claim

$$\left\{ \begin{array}{l} \{y_n^{(j)}\}_{n=n_0+1}^{\infty}, j = 0, 1, \quad \text{is a bounded, equicontinuous} \\ \text{family on} \quad \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]. \end{array} \right. \tag{2.24}$$

Firstly note

$$|y_n|_\infty \leq |y_{n_0}|_\infty \leq \sup_{[0,1]} \beta(t) = a_0 \quad \text{for} \quad t \in [0,1] \quad \text{and} \quad n \geq n_0 + 1. \tag{2.25}$$

Let

$$\epsilon = \min_{t \in \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]} \alpha(t).$$

Now (2.7) guarantees the existence of ψ_ϵ (as described in (2.7)) with

$$|f(t,y,z)| \leq \psi_\epsilon(|z|) \quad \text{for} \quad (t,y,z) \in (0,1) \times [\epsilon, a_0] \times \mathbf{R}.$$

This implies

$$|g_n(t, y_n(t), y'_n(t))| \leq \psi_\epsilon(|y'_n(t)|) \quad \text{for} \quad t \in [a,b] \equiv \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \subseteq e_{n_0}$$

and $n \geq n_0 + 1$. As a result

$$|y''_n(t)| \leq q(t)\, \psi_\epsilon(|y'_n(t)|) \quad \text{for} \quad t \in [a,b] \quad \text{and} \quad n \geq n_0 + 1. \tag{2.26}$$

The mean value theorem implies that there exists $\tau_{1,n} \in (a,b)$ with

$$|y'(\tau_{1,n})| = \frac{|y_(b) - y_n(a)|}{b-a} \leq \frac{2\, a_0}{b-a} = d_{n_0} \quad \text{for} \quad n \geq n_0.$$

Fix $n \geq n_0 + 1$ and let $t \in [a,b]$. Without loss of generality assume $y'_n(t) > d_{n_0}$. Then there exists $\tau_1 \in (a,b)$ with $y'_n(\tau_1) = d_{n_0}$ and $y'_n(s) > d_{n_0}$ for s between τ_1 and t. Without loss of generality assume $\tau_1 < s$. From (2.26) we have

$$\frac{y''_n(s)}{\psi_\epsilon(y'_n(s))} \leq q(s) \quad \text{for} \quad s \in (\tau_1, t),$$

so integration from τ_1 to t yields

$$\int_{d_{n_0}}^{y_n'(t)} \frac{du}{\psi_\epsilon(u)} \leq \int_0^1 q(s)\,ds.$$

Let $I_{n_0}(z) = \int_{d_{n_0}}^{z} \frac{du}{\psi_\epsilon(u)}$, so

$$|y_n'(t)| \leq I_{n_0}^{-1}\left(\int_0^1 q(s)\,ds\right) \equiv R_{n_0}.$$

A similar bound is obtained for the other cases, so

$$|y_n'(s)| \leq R_{n_0} \quad \text{for} \quad s \in [a,b] = \left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right] \tag{2.27}$$

and $n \geq n_0 + 1$. Now (2.25), (2.26) and (2.27) guarantee that (2.24) holds. The Arzela–Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $z_{n_0} \in C^1\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ with $y_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0}^{(j)}$ on $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\begin{cases} \{y_n^{(j)}\}_{n=n_0+2}^{\infty}, j = 0, 1, & \text{is a bounded, equicontinuous} \\ \text{family on} & \left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right], \end{cases} \tag{2.28}$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function

$$z_{n_0+1} \in C^1\left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right]$$

with $y_n^{(j)}$, $j = 0, 1$, converging uniformly to $z_{n_0+1}^{(j)}$ on $\left[\frac{1}{2^{n_0+2}}, 1 - \frac{1}{2^{n_0+2}}\right]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $\left[\frac{1}{2^{n_0+1}}, 1 - \frac{1}{2^{n_0+1}}\right]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \ldots \supseteq N_k \supseteq \ldots$$

and functions

$$z_k \in C^1\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$$

with

$$y_n^{(j)},\ j = 0,\ 1, \quad \text{converging uniformly to } z_k^{(j)} \text{ on } \left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$$

as $n \to \infty$ through N_k, and

$$z_k = z_{k-1} \text{ on } \left[\frac{1}{2^k}, 1 - \frac{1}{2^k}\right].$$

Define a function $y : [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $\left[\frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+1}}\right]$ and $y(0) = y(1) = 0$. Notice y is well defined and $\alpha(t) \le y(t) \le y_{n_0}(t) \le \beta(t)$ for $t \in (0,1)$. Next fix $t \in (0,1)$ (without loss of generality assume $t \neq \frac{1}{2}$) and let $m \in \{n_0, n_0+1, ...\}$ be such that $\frac{1}{2^{m+1}} < t < 1 - \frac{1}{2^{m+1}}$. Let $N_m^\star = \{n \in N_m : n \ge m\}$. Now y_n, $n \in N_m^\star$, satisfies the integral equation

$$\begin{aligned} y_n(x) &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^x (s-x)q(s)g_n(s, y_n(s), y_n'(s))\,ds \\ &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^x (s-x)q(s)f(s, y_n(s), y_n'(s))\,ds \end{aligned}$$

for $x \in \left[\frac{1}{2^{m+1}}, 1 - \frac{1}{2^{m+1}}\right]$. Let $n \to \infty$ through $N_m^\star$ to obtain

$$z_m(x) = z_m\left(\frac{1}{2}\right) + z_m'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \int_{\frac{1}{2}}^x (s-x)q(s)f(s, z_m(s), z_m'(s))\,ds,$$

so in particular

$$y(t) = y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^t (s-t)\,q(s)\,f(s, y(s), y'(s))\,ds.$$

We can do this argument for each $t \in (0,1)$, so $y''(t) + q(t)\,f(t, y(t), y'(t)) = 0$ for $t \in (0,1)$. It remains to show y is continuous at 0 and 1.

Let $\epsilon > 0$ be given. Now since $\lim_{n\to\infty} y_n(0) = 0$ there exists $n_1 \in \{n_0, n_0+1, ...\}$ with $y_{n_1}(0) < \frac{\epsilon}{2}$. Since $y_{n_1} \in C[0,1]$ there exists $\delta_{n_1} > 0$ with

$$y_{n_1}(t) < \frac{\epsilon}{2} \text{ for } t \in [0, \delta_{n_1}].$$

Now for $n \ge n_1$ we have, since $\{y_n(t)\}$ is nonincreasing for each $t \in [0,1]$,

$$\alpha(t) \le y_n(t) \le y_{n_1}(t) < \frac{\epsilon}{2} \text{ for } t \in [0, \delta_{n_1}].$$

Consequently

$$\alpha(t) \le y(t) \le \frac{\epsilon}{2} < \epsilon \text{ for } t \in (0, \delta_{n_1}]$$

and so y is continuous at 0. Similarly y is continuous at 1. As a result $y \in C[0,1]$. □

Suppose (2.2)–(2.5) hold and in addition assume the following conditions are satisfied:

$$\begin{cases} q(t)\, f(t, y, \alpha'(t)) + \alpha''(t) > 0 \ \text{ for} \\ (t, y) \in (0,1) \times \{\, y \in (0, \infty) : \ y < \alpha(t) \,\} \end{cases} \tag{2.29}$$

and

$$\begin{cases} \text{there exists a function } \ \beta \in C[0,1] \cap C^2(0,1) \\ \text{with } \ \beta(t) \geq \rho_{n_0} \ \text{ for } \ t \in [0,1] \ \text{ and with} \\ q(t)\, f(t, \beta(t), \beta'(t)) + \beta''(t) \leq 0 \ \text{ for } \ t \in (0,1) \ \text{ and} \\ q(t)\, f\left(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) + \beta''(t) \leq 0 \ \text{ for } \ t \in \left(0, \frac{1}{2^{n_0+1}}\right). \end{cases} \tag{2.30}$$

Also if (2.7) and (2.8) hold, then the result in Theorem 2.1 is again true. This follows immediately from Theorem 2.1 once we show (2.6) holds i.e. once we show $\beta(t) \geq \alpha(t)$ for $t \in [0,1]$. Suppose it is false. Then $\alpha - \beta$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$, so $(\alpha - \beta)'(\tau_0) = 0$ and $(\alpha - \beta)''(\tau_0) \leq 0$. Now $\alpha(\tau_0) > \beta(\tau_0)$ and (2.29) implies

$$q(\tau_0)\, f(\tau_0, \beta(\tau_0), \beta'(\tau_0)) + \alpha''(\tau_0) = q(\tau_0)\, f(\tau_0, \beta(\tau_0), \alpha'(\tau_0)) + \alpha''(\tau_0) \ > \ 0,$$

and this together with (2.30) yields

$$(\alpha - \beta)''(\tau_0) = \alpha''(\tau_0) - \beta''(\tau_0) \geq \alpha''(\tau_0) + q(\tau_0)\, f(\tau_0, \beta(\tau_0), \beta'(\tau_0)) \ > \ 0,$$

a contradiction. Thus we have

Corollary 2.2. *Let $n_0 \in \{1, 2, ...\}$ be fixed and suppose (2.2) – (2.5), (2.7), (2.8), (2.29) and (2.30) hold. Then (2.1) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [0,1]$.*

Remark 2.1. (i). If in (2.4) we replace $\frac{1}{2^{n+1}} \leq t \leq 1$ with $0 \leq t \leq 1 - \frac{1}{2^{n+1}}$ then one would replace (2.6) with

$$\begin{cases} \exists\ \beta \in C^1[0,1] \cap C^2(0,1) \ \text{ with } \ \beta(t) \geq \alpha(t), \ \beta(t) \geq \rho_{n_0} \\ \text{for } \ t \in [0,1] \ \text{ with } \ q(t)\, f(t, \beta(t), \beta'(t)) + \beta''(t) \leq 0 \\ \text{for } \ t \in (0,1) \ \text{ and } \ q(t)\, f\left(1 - \frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) + \beta''(t) \leq 0 \\ \text{for } \ t \in \left(1 - \frac{1}{2^{n_0+1}}, 1\right). \end{cases} \tag{2.31}$$

(ii). If in (2.4) we replace $\frac{1}{2^{n+1}} \leq t \leq 1$ with $\frac{1}{2^{n+1}} \leq t \leq 1 - \frac{1}{2^{n+1}}$ then one would replace (2.6) with

$$\begin{cases} \exists\ \beta \in C^1[0,1] \cap C^2(0,1) \ \text{ with } \ \beta(t) \geq \alpha(t), \ \beta(t) \geq \rho_{n_0} \\ \text{for } \ t \in [0,1] \ \text{ with } \ q(t)\, f(t, \beta(t), \beta'(t)) + \beta''(t) \leq 0 \\ \text{for } \ t \in (0,1) \ \text{ and } \ q(t)\, f\left(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) + \beta''(t) \leq 0 \\ \text{for } \ t \in \left(0, \frac{1}{2^{n_0+1}}\right), \ \ q(t)\, f\left(1 - \frac{1}{2^{n_0+1}}, \beta(t), \beta'(t)\right) + \beta''(t) \leq 0 \\ \text{for } \ t \in \left(1 - \frac{1}{2^{n_0+1}}, 1\right). \end{cases} \tag{2.32}$$

This is clear once one changes the definition of e_n and θ_n. For example in case (ii), take

$$e_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}\,,\, \min\left\{t, 1 - \frac{1}{2^{n+1}}\right\}\right\}.$$

Finally we discuss condition (2.5) and (2.29). Suppose the following condition is satisfied:

$$\text{(2.33)} \qquad \left\{\begin{array}{l} \text{let } n \in \{n_0, n_0+1, ...\} \text{ and associated with each } n \text{ we} \\ \text{have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and there exists a constant} \\ k_0 > 0 \text{ such that for } \frac{1}{2^{n+1}} \le t \le 1,\ 0 < y \le \rho_n \text{ and } z \in \mathbf{R} \\ \text{we have } q(t)\, f(t,y,z) \ge k_0. \end{array}\right.$$

We will show if (2.33) holds then (2.5) (and of course (2.4)) and (2.29) are satisfied (we also note that $\frac{1}{2^{n+1}} \le t \le 1$ in (2.33) could be replaced by $0 \le t \le 1 - \frac{1}{2^{n+1}}$ (respectively $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$) and (2.5), (2.29) hold with $\frac{1}{2^{n+1}} \le t \le 1$ replaced by $0 \le t \le 1 - \frac{1}{2^{n+1}}$ (respectively $\frac{1}{2^{n+1}} \le t \le 1 - \frac{1}{2^{n+1}}$)).

To show (2.5) and (2.29) recall the following Lemma from [5].

Lemma 2.3. *Let e_n be as described in Theorem 2.1 (or Remark 2.1) and let $0 < \epsilon_n < 1$ with $\epsilon_n \downarrow 0$. Then there exists $\lambda \in C^2[0,1]$ with $\sup_{[0,1]} |\lambda''(t)| > 0$ and $\lambda(0) = \lambda(1) = 0$ with*

$$0 < \lambda(t) \le \epsilon_n \quad \textit{for} \quad t \in e_n \backslash e_{n-1},\ \ n \ge 1.$$

Let $\epsilon_n = \rho_n$ (and $n \ge n_0$) and let λ be as in Lemma 2.3. From (2.33) there exists $k_0 > 0$ with

$$\text{(2.34)} \qquad \left\{\begin{array}{l} q(t)\, f(t,y,z) \ge k_0 \quad \text{for} \\ (t,y,z) \in (0,1) \times \{y \in (0,\infty) :\ y \le \lambda(t)\} \times \mathbf{R}, \end{array}\right.$$

since if $t \in e_n \setminus e_{n-1}$ $(n \ge n_0)$ then $y \le \lambda(t)$ implies $y \le \rho_n$. Let

$$M = \sup_{[0,1]} |\lambda''(t)|,\ m = \min\left\{1\,,\, \frac{k_0}{M+1}\right\} \quad \text{and} \quad \alpha(t) = m\,\lambda(t),\ t \in [0,1].$$

In particular since $\alpha(t) \le \lambda(t)$ we have from (2.34) that

$$q(t)\, f(t, \alpha(t), z) + \alpha''(t) \ge k_0 + \alpha''(t) \ge k_0 - \frac{k_0\, |\lambda''(t)|}{M+1} > 0$$

for $(t,z) \in (0,1) \times \mathbf{R}$, and also

$$q(t)\, f(t,y,\alpha'(t)) + \alpha''(t) \geq k_0 + \alpha''(t) > 0$$

for $(t,y) \in (0,1) \times \{y \in (0,\infty) : y \leq \alpha(t)\}$. Thus (2.5) and (2.29) hold.

Theorem 2.4. *Let $n_0 \in \{1,2,...\}$ be fixed and suppose* (2.2), (2.3), (2.7), (2.8), (2.30) *and* (2.33) *hold. Then* (2.1) *has a solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.*

Example. Consider the boundary value problem

$$\text{(2.35)} \qquad \begin{cases} y'' + \frac{t}{y^2} + |y'|^\alpha - \mu^2 = 0, \;\; 0 < t < 1 \\ y(0) = y(1) = 0 \end{cases}$$

with $\mu > 0$ and $0 \leq \alpha \leq 1$. Then (2.35) has a solution $y \in C[0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1)$.

To see that (2.35) has the desired solution we will apply Theorem 2.4 with $q = 1$, $f(t,y,z) = \frac{t}{y^2} + |z|^\alpha - \mu^2$ and

$$\rho_n = \left(\frac{1}{2^{n+1}\,(\mu^2+1)} \right)^{\frac{1}{2}}, \quad k_0 = 1 \;\text{ and }\; n_0 = 1.$$

Clearly (2.2) and (2.3) hold and notice also if $n \in \{1,2,...\}$, $\frac{1}{2^{n+1}} \leq t \leq 1$, $0 < y \leq \rho_n$ and $z \in \mathbf{R}$ we have

$$q(t)\, f(t,y,z) \geq \frac{t}{\rho_n^2} - \mu^2 \geq \frac{1}{2^{n+1}\,\rho_n^2} - \mu^2 = (\mu^2+1) - \mu^2 = 1,$$

so (2.33) is also true. Next let $\beta(t) = M + \rho_1$ where M is chosen large enough so that

$$\frac{1}{(M+\rho_1)^2} \leq \mu^2.$$

Notice (2.30) is immediate since

$$q(t)\, f(t,\beta(t),\beta'(t)) + \beta''(t) = \frac{t}{[\beta(t)]^2} - \mu^2 \leq \frac{1}{(M+\rho_1)^2} - \mu^2 \leq 0$$

for $t \in (0,1)$, and

$$q(t)\, f\left(\frac{1}{2^{n_0+1}}, \beta(t), \beta'(t) \right) + \beta''(t) = \frac{1}{4\,[\beta(t)]^2} - \mu^2 \leq \frac{1}{(M+\rho_1)^2} - \mu^2 \leq 0$$

for $t \in \left(0, \frac{1}{4}\right)$. Next let

$$\psi_\epsilon(z) = \frac{1}{\epsilon^2} + \mu^2 + z^\alpha$$

and notice (2.7) and (2.8) are satisfied since

$$|f(t,y,z)| \leq \frac{1}{\epsilon^2} + \mu^2 + |z|^\alpha = \psi_\epsilon(|z|) \quad \text{for} \quad t \in (0,1),\ y \geq \epsilon,\ z \in \mathbf{R}$$

and

$$\int_0^\infty \frac{du}{\psi_\epsilon(u)} = \infty \quad \text{since} \quad 0 \leq \alpha \leq 1.$$

Existence now follows from Theorem 2.4.

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Fractional Opial Inequalities for Several Functions with Applications

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Key Words and Phrases: Opial type inequality, fractional derivative, system of fractional differential equations, uniqueness of solution, upper bound of solution.

Abstract. A large variety of very general L_p $(1 \leq p \leq \infty)$ form Opial type inequalities ([10]) is presented involving generalized fractional derivatives ([5], [8]) of several functions in different orders and powers. The above are based on a generalization of Taylor's formula for generalized fractional derivatives ([5]). From the established results derive several other particular results of special interest. Applications of some of these special inequalities are given in proving uniqueness of solution and in giving upper bounds to solutions of initial value problems involving a very general system of several fractional differential equations. Upper bounds to various fractional derivatives of the solutions that are involved in the above systems are given too.

0 Introduction

Opial inequalities appeared for the first time in [10] and then many authors dealt with them in different directions and for various cases. For a complete

recent account on the activity of this field see [3], and still it remains a very active area of research. One of their main attractions to these inequalities is their applications, especially to proving uniqueness and upper bounds of solution of initial value problems in differential equations. The author was the first to present Opial inequalities involving fractional derivatives of functions in [4], [5] with applications to fractional differential equations.

Fractional derivatives come up naturally in a number of fields, especially in Physics, see the recent book [9]. To name a few topics such as, fractional Kinetics of Hamiltonian Chaotic systems, Polymer Physics and Rheology, Regular variation in Thermodynamics, Biophysics, fractional time evolution, fractal time series, etc. One there deals also with stochastic fractional-difference equations and fractional diffusion equations. Great applications of these can be found in the study of DNA sequences. Other fractional differential equations arise in the study of suspensions, coming from the fluid dynamical modeling of certain blood flow phenomena. An excellent account in the study of fractional differential equations is in the recent book [11].

The study of fractional calculus started from 1695 by L'Hospital and Leibniz, also continued later by J. Fourier in 1822 and Abel in 1823, and continuous to our days in an increased fashion due to its many applications and necessity to deal with fractional phenomena and structures.

In this paper the author is greatly motivated and inspired by the very important papers [1], [2]. Of course there the authors are dealing with other kinds of derivative. Here the author continues his study of fractional Opial inequalities now involving several different functions and produces a wide variety of corresponding results with important applications to systems of several fractional differential equations. This article is a generalization of the author's earlier article [6].

We start in Section 1 with Preliminaries, we continue in Section 2 with the main results and we finish in Section 3 with applications.

To give an idea to the reader of the kind of inequalities we are dealing with, briefly we mention a simple one

$$\int_a^x \left(\sum_{j=1}^{M} \left| (D_a^{\gamma} f_j)(w) \right| \left| (D_a^{\nu} f_j)(w) \right| \right) dw$$

$$\leq \left(\frac{(x-a)^{\nu-\gamma}}{2\Gamma(\nu-\gamma)\sqrt{\nu-\gamma}\sqrt{2\nu-2\gamma-1}} \right) \left\{ \int_a^x \left(\sum_{j=1}^{M} \left((D_a^{\nu} f_j)(w) \right)^2 \right) dw \right\}, \quad (*)$$

all $a \leq x \leq b$, for certain kind of continuous functions f_j, $j = 1, \ldots, M \in \mathbb{N}$; $\gamma, \nu \geq 1$, $\nu - \gamma \geq 1$, etc. Furthermore one system of fractional differential equations we are dealing with briefly is of the form

$$(D_a^{\nu} f_j)(t) = F_j\big(t, \{(D_a^{\gamma_i} f_1)(t)\}_{i=1}^r, \{(D_a^{\gamma_i} f_2)(t)\}_{i=1}^r,$$

$$\ldots, \{(D_a^{\gamma_i} f_M)(t)\}_{i=1}^r\big), \quad \text{all } t \in [a,b], \quad (**)$$

for $j = 1, 2, \ldots, M \in \mathbb{N}$ and with $f_j^{(i)}(a) = a_{ij} \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, where $n := [\nu]$, $\nu \geq 2$, etc.

In the literature there are many different definitions of fractional derivatives, some of them being equivalent, see [9], [11]. In this article we use one of the most recent due to J. Canavati [8], generalized in [4] and [5] by the author.

One of the advantages of Canavati fractional derivatives is that in applications to fractional initial value problems we need only n initial conditions, like with the ordinary derivative case, while with other definitions of fractional derivatives we need $n+1$ or more initial conditions, see [11].

1 Preliminaries

In the next we follow [8]. Let $g \in C([0,1])$. Let ν be a positive number, $n := [\nu]$ and $\alpha := \nu - n$ $(0 < \alpha < 1)$. Define

$$(J_\nu g)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} g(t)\, dt, \quad 0 \leq x \leq 1, \tag{1}$$

the *Riemann–Liouville integral*, where Γ is the gamma function. We define the subspace $C^\nu([0,1])$ of $C^n([0,1])$ as follows:

$$C^\nu([0,1]) := \{g \in C^n([0,1]) \colon J_{1-\alpha} D^n g \in C^1([0,1])\},$$

where $D := \frac{d}{dx}$. So for $g \in C^\nu([0,1])$, we define the *ν-fractional derivative* of g as

$$D^\nu g := D J_{1-\alpha} D^n g. \tag{2}$$

When $\nu \geq 1$ we have the Taylor's formula

$$g(t) = g(0) + g'(0)t + g''(0)\frac{t^2}{2!} + \cdots + g^{(n-1)}(0)\frac{t^{n-1}}{(n-1)!} + (J_\nu D^\nu g)(t), \quad \text{for all } t \in [0,1]. \tag{3}$$

When $0 < \nu < 1$ we find

$$g(t) = (J_\nu D^\nu g)(t), \quad \text{for all } t \in [0,1]. \tag{4}$$

Next we transfer above notions over to arbitrary $[a,b] \subseteq \mathbb{R}$ (see [5]). Let $x, x_0 \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a,b])$ and define

$$(J_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t)\, dt, \quad x_0 \leq x \leq b, \tag{5}$$

the *generalized Riemann–Liouville integral*. We define the subspace $C_{x_0}^\nu([a,b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \{f \in C^n([a,b]) \colon J_{1-\alpha}^{x_0} D^n f \in C^1([x_0,b])\}.$$

For $f \in C^{\nu}_{x_0}([a,b])$, we define the *generalized ν-fractional derivative of f over* $[x_0,b]$ as

$$D^{\nu}_{x_0} f := D J^{x_0}_{1-\alpha} f^{(n)} \quad (f^{(n)} := D^n f). \tag{6}$$

Observe that

$$(J^{x_0}_{1-\alpha} f^{(n)})(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t)\, dt$$

exists for $f \in C^{\nu}_{x_0}([a,b])$.

We recall the following generalization of Taylor's formula (see [8], [5]).

Theorem 1. *Let $f \in C^{\nu}_{x_0}([a,b])$, $x_0 \in [a,b]$, fixed.*

(i) *If $\nu \geq 1$ then*

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} \\ + \cdots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} \\ + (J^{x_0}_{\nu} D^{\nu}_{x_0} f)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \end{aligned} \tag{7}$$

(ii) *If $0 < \nu < 1$ then*

$$f(x) = (J^{x_0}_{\nu} D^{\nu}_{x_0} f)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \tag{8}$$

We make

Remark 1. 1) $(D^n_{x_0} f) = f^{(n)}$, $n \in \mathbb{N}$.

2) Let $f \in C^{\nu}_{x_0}([a,b])$, $\nu \geq 1$ and $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$; $n := [\nu]$. Then by (7)

$$f(x) = (J^{x_0}_{\nu} D^{\nu}_{x_0} f)(x).$$

I.e.

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} (D^{\nu}_{x_0} f)(t)\, dt, \tag{9}$$

for all $x \in [a,b]$ with $x \geq x_0$. Notice that (9) is true, also when $0 < \nu < 1$.

We also make

Remark 2. Let $\nu, \gamma \geq 1$ such that $\nu - \gamma \geq 1$, so that $\gamma < \nu$. Call $n := [\nu]$, $\alpha := \nu - n$; $m := [\gamma]$, $\rho := \gamma - m$. Note that $\nu - m \geq 1$ and $n - m \geq 1$. Let $f \in C^{\nu}_{x_0}([a,b])$ be such that $f^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$. Hence by (7)

$$f(x) = (J^{x_0}_{\nu} D^{\nu}_{x_0} f)(x), \quad \text{for all } x \in [a,b] : x \geq x_0.$$

Therefore by Leibnitz's formula and $\Gamma(p+1) = p\Gamma(p)$, $p > 0$, we get that

$$f^{(m)}(x) = (J^{x_0}_{\nu-m} D^{\nu}_{x_0} f)(x), \quad \text{for all } x \geq x_0.$$

It follows that $f \in C^{\gamma}_{x_0}([a,b])$ and thus $(D^{\gamma}_{x_0} f)(x) := (DJ^{x_0}_{1-\rho} f^{(m)})(x)$ exists for all $x \geq x_0$.

Easily we obtain

$$(D^{\gamma}_{x_0} f)(x) = D((J^{x_0}_{1-\rho} f^{(m)})(x)) = \frac{1}{\Gamma(\nu-\gamma)} \int_{x_0}^{x} (x-t)^{(\nu-\gamma)-1} (D^{\nu}_{x_0} f)(t)\, dt, \tag{10}$$

and thus

$$(D^{\gamma}_{x_0} f)(x) = (J^{x_0}_{\nu-\gamma}(D^{\nu}_{x_0} f))(x)$$

and is continuous in x on $[x_0, b]$.

2 Main Results

Here we use a lot the following basic inequalities.

Let $a_1, \ldots, a_n \geq 0$, $n \in \mathbb{N}$, then

$$a_1^r + \cdots + a_n^r \leq (a_1 + \cdots + a_n)^r, \quad r \geq 1, \tag{11}$$

and

$$a_1^r + \cdots + a_n^r \leq n^{1-r}(a_1 + \cdots + a_n)^r, \quad 0 \leq r \leq 1. \tag{12}$$

Our first result follows next

Theorem 2. *Let $\nu, \gamma_1, \gamma_2 \geq 1$ such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$ and $f_j \in C^{\nu}_{x_0}([a,b])$ with $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. Here $x, x_0 \in [a,b]: x \geq x_0$. Consider also $p(t) > 0$, and $q(t) \geq 0$ continuous functions on $[x_0, b]$. Let $\lambda_\nu > 0$ and $\lambda_\alpha, \lambda_\beta \geq 0$ such that $\lambda_\nu < p$, where $p > 1$. Set*

$$P_k(w) := \int_{x_0}^{w} (w-t)^{\frac{(\nu-\gamma_k-1)p}{p-1}} (p(t))^{-\frac{1}{p-1}}\, dt, \quad k = 1, 2,\ x_0 \leq w \leq b; \tag{13}$$

$$A(w) := \frac{q(w) \cdot (P_1(w))^{\lambda_\alpha \left(\frac{p-1}{p}\right)} \cdot (P_2(w))^{\lambda_\beta \left(\frac{p-1}{p}\right)} (p(w))^{-\frac{\lambda_\nu}{p}}}{(\Gamma(\nu-\gamma_1))^{\lambda_\alpha} \cdot (\Gamma(\nu-\gamma_2))^{\lambda_\beta}}; \tag{14}$$

$$A_0(x) := \left(\int_{x_0}^{x} A(w)^{\frac{p}{p-\lambda_\nu}}\, dw \right)^{\frac{p-\lambda_\nu}{p}}. \tag{15}$$

Call

$$\varphi_1(x) := \left(A_0(x) \big|_{\lambda_\beta = 0} \right) \cdot \left(\frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\frac{\lambda_\nu}{p}}, \tag{16}$$

$$\delta_1^* := \begin{cases} M^{1-\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 2^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)-1}, & \text{if } \lambda_\alpha + \lambda_\nu \geq p. \end{cases} \tag{17}$$

If $\lambda_\beta = 0$, we obtain that

$$\int_{x_0}^{x} q(w) \left(\sum_{j=1}^{M} |(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu} \right) dw$$

$$\leq \delta_1^* \cdot \varphi_1(x) \cdot \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} |(D_{x_0}^{\nu} f_j)(w)|^p \right) dw \right]^{\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)}, \quad (18)$$

all $x_0 \leq x \leq b$.

Proof. From Theorem 2 of [6] we get

$$\int_{x_0}^{x} q(w) \big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu}$$

$$+ |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_{j+1})(w)|^{\lambda_\nu} \big] dw \quad (19)$$

$$\leq \delta_1 \varphi_1(x) \left[\int_{x_0}^{x} p(w) \big[|(D_{x_0}^{\nu} f_j)(w)|^p + |(D_{x_0}^{\nu} f_{j+1})(w)|^p \big] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)},$$

$j = 1, 2, \ldots, M-1$, where

$$\delta_1 := \begin{cases} 2^{1-\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 1, & \text{if } \lambda_\alpha + \lambda_\nu \geq p. \end{cases} \quad (20)$$

Hence by adding all the above we find

$$\int_{x_0}^{x} q(w) \Bigg(\sum_{j=1}^{M-1} \big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu}$$

$$+ |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_{j+1})(w)|^{\lambda_\nu} \big] \Bigg) dw \quad (21)$$

$$\leq \delta_1 \varphi_1(x) \cdot \left(\sum_{j=1}^{M-1} \left[\int_{x_0}^{x} p(w) [|(D_{x_0}^{\nu} f_j)(w)|^p + |(D_{x_0}^{\nu} f_{j+1})(w)|^p] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)} \right).$$

Also it holds

$$\int_{x_0}^{x} q(w) \big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_1)(w)|^{\lambda_\nu}$$

$$+ |(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_M)(w)|^{\lambda_\nu} \big] dw \quad (22)$$

$$\leq \delta_1 \varphi_1(x) \left[\int_{x_0}^{x} p(w) \big[|(D_{x_0}^{\nu} f_1)(w)|^p + |(D_{x_0}^{\nu} f_M)(w)|^p \big] dw \right]^{\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)}.$$

Call

$$\varepsilon_1 = \begin{cases} 1, & \text{if } \lambda_\alpha + \lambda_\nu \geq p \\ M^{1-\left(\frac{\lambda_\alpha + \lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p. \end{cases} \quad (23)$$

Adding (21) and (22), and using (11) and (12) we have

$$
\begin{aligned}
&2\int_{x_0}^{x} q(w)\left(\sum_{j=1}^{M}|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu}\right)dw \\
&\leq\ \delta_1\varphi_1(x)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Bigg[\int_{x_0}^{x}p(w)[|(D_{x_0}^{\nu}f_j)(w)|^p \\
&\quad+|(D_{x_0}^{\nu}f_{j+1})(w)|^p]dw\Bigg]^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}\Bigg\} \\
&\quad+\left\{\int_{x_0}^{x}p(w)[|(D_{x_0}^{\nu}f_1)(w)|^p+|(D_{x_0}^{\nu}f_M)(w)|^p]dw\right\}^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}\Bigg\} \\
&\leq\ \delta_1\varepsilon_1\varphi_1(x)\left\{\int_{x_0}^{x}p(w)\left[2\sum_{j=1}^{M}|(D_{x_0}^{\nu}f_j)(w)|^p\right]dw\right\}^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}.
\end{aligned}
$$

We have proved

$$
\begin{aligned}
&\int_{x_0}^{x} q(w)\left(\sum_{j=1}^{M}|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu}\right)dw \\
&\quad\leq\ \delta_1\left(2^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)-1}\right)\varepsilon_1\varphi_1(x) \\
&\qquad\cdot\left\{\int_{x_0}^{x}p(w)\left[\sum_{j=1}^{M}|(D_{x_0}^{\nu}f_j)(w)|^p\right]dw\right\}^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)}. \qquad (24)
\end{aligned}
$$

Clearly here we have

$$
\delta_1^{*}=\delta_1\left(2^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)-1}\right)\varepsilon_1. \qquad (25)
$$

From (24) and (25) we derive (18). ■

Next we give

Theorem 3. *All here as in Theorem 2. Denote*

$$
\delta_3:=\begin{cases}2^{\frac{\lambda_\beta}{\lambda_\nu}}-1, & \text{if } \lambda_\beta\geq\lambda_\nu,\\ 1, & \text{if } \lambda_\beta\leq\lambda_\nu,\end{cases} \qquad (26)
$$

$$
\varepsilon_2:=\begin{cases}1, & \text{if } \lambda_\nu+\lambda_\beta\geq p,\\ M^{1-\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}, & \text{if } \lambda_\nu+\lambda_\beta\leq p,\end{cases} \qquad (27)
$$

and

$$
\varphi_2(x):=\left(A_0(x)\big|_{\lambda_\alpha=0}\right)2^{\left(\frac{p-\lambda_\nu}{p}\right)}\left(\frac{\lambda_\nu}{\lambda_\beta+\lambda_\nu}\right)^{\frac{\lambda_\nu}{p}}\delta_3^{\frac{\lambda_\nu}{p}}. \qquad (28)
$$

If $\lambda_\alpha = 0$, then it holds

$$\int_{x_0}^{x} q(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\big[|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\big]\Bigg\} + \big[|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_1)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\big]\Bigg\}dw$$
$$\leq 2^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}\varepsilon_2\varphi_2(x)\cdot\Bigg\{\int_{x_0}^{x} p(w)\cdot\Bigg[\sum_{j=1}^{M}|(D_{x_0}^{\nu}f_j)(w)|^p\Bigg]dw\Bigg\}^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)},\quad x\geq x_0. \tag{29}$$

Proof. From Theorem 3 of [6] we have

$$\int_{x_0}^{x} q(w)\big[|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\big]\,dw$$
$$\leq\ \varphi_2(x)\Bigg(\int_{x_0}^{x} p(w)\big[|(D_{x_0}^{\nu}f_j)(w)|^p + |(D_{x_0}^{\nu}f_{j+1})(w)|^p\big]dw\Bigg)^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}, \tag{30}$$

for $j = 1,\ldots,M-1$. Hence by adding all the above we get

$$\int_{x_0}^{x} q(w)\Bigg(\sum_{j=1}^{M-1}\big[|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\big]\Bigg)dw$$
$$\leq\ \varphi_2(x)\Bigg\{\sum_{j=1}^{M-1}\Bigg(\int_{x_0}^{x} p(w)[|(D_{x_0}^{\nu}f_j)(w)|^p + |(D_{x_0}^{\nu}f_{j+1})(w)|^p\,dw\Bigg)^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}\Bigg\}. \tag{31}$$

Similarly it holds

$$\int_{x_0}^{x} q(w)\big[|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_1)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\big]\,dw$$
$$\leq\ \varphi_2(x)\Bigg(\int_{x_0}^{x} p(w)\big[|(D_{x_0}^{\nu}f_1)(w)|^p + |(D_{x_0}^{\nu}f_M)(w)|^p\big]\,dw\Bigg)^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}. \tag{32}$$

Adding (31) and (32) and using (11), (12) we produce (29). ■

It follows the general case

Theorem 4. *All here as in Theorem 2. Denote*

$$\tilde{\gamma}_1 := \begin{cases} 2^{\left(\frac{\lambda_\alpha+\lambda_\beta}{\lambda_\nu}\right)} - 1, & \text{if } \lambda_\alpha+\lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\alpha+\lambda_\beta \leq \lambda_\nu, \end{cases} \tag{33}$$

and

$$\tilde{\gamma}_2 := \begin{cases} 1, & \text{if } \lambda_\alpha+\lambda_\beta+\lambda_\nu \geq p, \\ 2^{1-\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha+\lambda_\beta+\lambda_\nu \leq p. \end{cases} \tag{34}$$

Set

$$\begin{aligned} \varphi_3(x) := A_0(x) \cdot \left(\frac{\lambda_\nu}{(\lambda_\alpha+\lambda_\beta)(\lambda_\alpha+\lambda_\beta+\lambda_\nu)}\right)^{\frac{\lambda_\nu}{p}} \\ \cdot \left[\lambda_\alpha^{\frac{\lambda_\nu}{p}}\tilde{\gamma}_2 + 2^{\left(\frac{p-\lambda_\nu}{p}\right)}(\tilde{\gamma}_1\lambda_\beta)^{\frac{\lambda_\nu}{p}}\right], \end{aligned} \tag{35}$$

and

$$\varepsilon_3 := \begin{cases} 1, & \text{if } \lambda_\alpha+\lambda_\beta+\lambda_\nu \geq p, \\ M^{1-\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, & \text{if } \lambda_\alpha+\lambda_\beta+\lambda_\nu \leq p. \end{cases} \tag{36}$$

Then it holds

$$\begin{aligned} &\int_{x_0}^{x} q(w)\Bigg[\sum_{j=1}^{M-1}\big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu} \\ &\quad + |(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_{j+1})(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\big] \\ &\quad + \big[|(D_{x_0}^{\gamma_1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_1)(w)|^{\lambda_\nu} \\ &\quad + |(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_M)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\big]\Bigg]\,dw \\ &\leq 2^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}\varepsilon_3\varphi_3(x)\cdot\left\{\int_{x_0}^{x} p(w)\left[\sum_{j=1}^{M}|(D_{x_0}^{\nu}f_j)(w)|^p\right]dw\right\}^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, \end{aligned} \tag{37}$$

all $x_0 \leq x \leq b$.

Proof. From Theorem 4 of [6] and adding altogether we have

$$\begin{aligned} &\sum_{j=1}^{M-1}\int_{x_0}^{x} q(w)\big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu} \\ &\quad + |(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_{j+1})(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\big]\,dw \\ &\leq \varphi_3(x)\sum_{j=1}^{M-1}\left(\int_{x_0}^{x} p(w)\big(|(D_{x_0}^{\nu}f_j)(w)|^p + |(D_{x_0}^{\nu}f_{j+1})(w)|^p\big)\,dw\right)^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)}, \end{aligned} \tag{38}$$

all $x_0 \le x \le b$.

Also it holds

$$\int_{x_0}^{x} q(w)\big[|(D_{x_0}^{\gamma_1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_1)(w)|^{\lambda_\nu} + |(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_M)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\big]\,dw \tag{39}$$
$$\le \varphi_3(x)\left(\int_{x_0}^{x} p(w)(|(D_{x_0}^{\nu}f_1)(w)|^p + |(D_{x_0}^{\nu}f_M)(w)|^p)\,dw\right)^{\left(\frac{\lambda_\alpha+\lambda_\beta+\lambda_\nu}{p}\right)},$$

all $x_0 \le x \le b$.

Adding (38) and (39), along with (11) and (12) we produce (37). ■

We continue with

Theorem 5. *Let $\nu \ge 3$ and $\gamma_1 \ge 1$ such that $\nu - \gamma_1 \ge 2$. Let $f_j \in C_{x_0}^{\nu}([a,b])$ with $f_j^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n-1$, $n := [\nu]$, $j = 1,\ldots,M \in \mathbb{N}$. Here $x, x_0 \in [a,b]: x \ge x_0$. Consider also $p(t) > 0$, and $q(t) \ge 0$ continuous functions on $[x_0,b]$. Let $\lambda_\alpha \ge 0$, $0 < \lambda_{\alpha+1} < 1$, and $p > 1$. Denote*

$$\theta_3 := \begin{cases} 2^{\left(\frac{\lambda_\alpha}{\lambda_{\alpha+1}}\right)} - 1, & \text{if } \lambda_\alpha \ge \lambda_{\alpha+1}, \\ 1, & \text{if } \lambda_\alpha \le \lambda_{\alpha+1} \end{cases} \tag{40}$$

$$L(x) := \left(2\int_{x_0}^{x} (q(w))^{\left(\frac{1}{1-\lambda_{\alpha+1}}\right)}\,dw\right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3\lambda_{\alpha+1}}{\lambda_\alpha+\lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{41}$$

and

$$P_1(x) := \int_{x_0}^{x} (x-t)^{\frac{(\nu-\gamma_1-1)p}{p-1}} (p(t))^{-\frac{1}{p-1}}\,dt, \tag{42}$$

$$T(x) := L(x)\cdot\left(\frac{P_1(x)^{\left(\frac{p-1}{p}\right)}}{\Gamma(\nu-\gamma_1)}\right)^{(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{43}$$

and

$$\omega_1 := \begin{cases} 2^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \le p, \\ 1, & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \ge p \end{cases}, \tag{44}$$

$$\Phi(x) := T(x)\omega_1. \tag{45}$$

Also put

$$\varepsilon_4 := \begin{cases} 1, & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \ge p, \\ M^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, & \text{if } \lambda_\alpha+\lambda_{\alpha+1} \le p \end{cases}. \tag{46}$$

Then it holds

$$\int_{x_0}^{x} q(w)\left\{\left\{\sum_{j=1}^{M-1}\big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1+1}f_{j+1})(w)|^{\lambda_{\alpha+1}}\right.\right.$$

$$+ |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_j)(w)|^{\lambda_{\alpha+1}} \Big] \Big\}$$
$$+ \Big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_M)(w)|^{\lambda_{\alpha+1}}$$
$$+ |(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_1)(w)|^{\lambda_{\alpha+1}} \Big] \Big\} dw$$
$$\leq 2^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)} \varepsilon_4 \Phi(x) \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} |(D_{x_0}^{\nu} f_j)(w)|^p \right) dw \right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, \quad (47)$$

all $x_0 \leq x \leq b$.

Proof. From Theorem 5 ([6]) we get

$$\int_{x_0}^{x} q(w) \sum_{j=1}^{M-1} \Big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_{j+1})(w)|^{\lambda_{\alpha+1}}$$
$$+ |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_j)(w)|^{\lambda_{\alpha+1}} \Big] dw$$
$$\leq \Phi(x) \sum_{j=1}^{M-1} \left[\int_{x_0}^{x} p(w) \big(|(D_{x_0}^{\nu} f_j)(w)|^p + |(D_{x_0}^{\nu} f_{j+1})(w)|^p \big) dw \right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)} \quad (48)$$

all $x_0 \leq x \leq b$.

Also it holds

$$\int_{x_0}^{x} q(w) \big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_M)(w)|^{\lambda_{\alpha+1}}$$
$$+ |(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_1+1} f_1)(w)|^{\lambda_{\alpha+1}} \big] dw$$
$$\leq \Phi(x) \left[\int_{x_0}^{x} p(w) (|(D_{x_0}^{\nu} f_1)(w)|^p + |(D_{x_0}^{\nu} f_M)(w)|^p \big] dw \right]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, \quad (49)$$

all $x_0 \leq x \leq b$. Adding (48) and (49), along with (11) and (12) we derive (47). ■

Next it comes

Theorem 6. *All here as in Theorem 2. Consider the special case* $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. *Denote*

$$\tilde{T}(x) := A_0(x) \left(\frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\frac{\lambda_\nu}{p}} 2^{\left(\frac{p-2\lambda_\alpha-3\lambda_\nu}{p}\right)}, \quad (50)$$

$$\varepsilon_5 := \left\{ \begin{array}{ll} 1, & \text{if } 2(\lambda_\alpha + \lambda_\nu) \geq p, \\ M^{1-\left(\frac{2(\lambda_\alpha+\lambda_\nu)}{p}\right)}, & \text{if } 2(\lambda_\alpha + \lambda_\nu) \leq p \end{array} \right\}. \quad (51)$$

Then it holds

$$\int_{x_0}^{x} q(w) \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_2} f_{j+1})(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu}$$

$$+ |(D_{x_0}^{\gamma_2} f_j)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_{j+1})(w)|^{\lambda_\nu} \Big] \Bigg\}$$

$$+ \Big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_2} f_M)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\nu} f_1)(w)|^{\lambda_\nu}$$

$$+ |(D_{x_0}^{\gamma_2} f_1)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_M)(w)|^{\lambda_\nu} \Big] \Bigg\} dw$$

$$\leq 2^{2\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)} \varepsilon_5 \tilde{T}(x) \left[\int_{x_0}^{x} p(w) \left(\sum_{j=1}^{M} |(D_{x_0}^{\nu} f_j)(w)|^p \right) dw \right]^{\left(2\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)\right)}, \quad (52)$$

all $x_0 \leq x \leq b$.

Proof. Based on Theorem 6 ([6]). The rest as in the proof of Theorem 5. ■

Next we give special cases of the above theorems.

Corollary 1 (to Theorem 2; $\lambda_\beta = 0$, $p(t) = q(t) = 1$). *It holds*

$$\int_{x_0}^{x} \left(\sum_{j=1}^{M} |(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu} \right) dw$$

$$\leq \delta_1^* \varphi_1(x) \left[\int_{x_0}^{x} \left[\sum_{j=1}^{M} |(D_{x_0}^{\nu} f_j)(w)|^p \right] dw \right]^{\left(\frac{\lambda_\alpha+\lambda_\nu}{p}\right)} \quad (53)$$

all $x_0 \leq x \leq b$.

In (53) $\left(A_0(x)\big|_{\lambda_\beta=0}\right)$ of $\varphi_1(x)$ is given in [6], Corollary 1, by equation (55).

Corollary 2 (to Theorem 2; $\lambda_\beta = 0$, $p(t) = q(t) = 1$, $\lambda_\alpha = \lambda_\nu = 1$, $p = 2$). *In detail*:

Let $\nu, \gamma_1 \geq 1$ *such that* $\nu - \gamma_1 \geq 1$, $f_j \in C_{x_0}^{\nu}([a,b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here* $x, x_0 \in [a,b]$: $x \geq x_0$. *Then it holds*

$$\int_{x_0}^{x} \left(\sum_{j=1}^{M} |(D_{x_0}^{\gamma_1} f_j)(w)| \, |(D_{x_0}^{\nu} f_j)(w)| \right) dw \quad (54)$$

$$\leq \left(\frac{(x-x_0)^{\nu-\gamma_1}}{2\Gamma(\nu-\gamma_1)\sqrt{\nu-\gamma_1}\sqrt{2\nu-2\gamma_1-1}} \right) \cdot \left\{ \int_{x_0}^{x} \left[\sum_{j=1}^{M} ((D_{x_0}^{\nu} f_j)(w))^2 \right] dw \right\},$$

all $x_0 \leq x \leq b$.

Proof. Based on our Corollary 1 and Corollary 1 of [6], especially equation (55) there. ■

Corollary 3 (to Theorem 3, $\lambda_\alpha = 0$, $p(t) = q(t) = 1$). *It holds*

$$\int_{x_0}^{x} \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[|(D_{x_0}^{\gamma_2} f_{j+1})(w)|^{\lambda_\beta} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu}$$

$$
\begin{aligned}
&+ |(D^{\gamma_2}_{x_0} f_j)(w)|^{\lambda_\beta} |(D^{\nu}_{x_0} f_{j+1})(w)|^{\lambda_\nu} \Big] \Bigg\} \\
&+ \Big[|(D^{\gamma_2}_{x_0} f_M)(w)|^{\lambda_\beta} |(D^{\nu}_{x_0} f_1)(w)|^{\lambda_\nu} \\
&+ |(D^{\gamma_2}_{x_0} f_1)(w)|^{\lambda_\beta} |(D^{\nu}_{x_0} f_M)(w)|^{\lambda_\nu} \Big] \Bigg\} dw \\
\leq\ & 2^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)} \varepsilon_2 \varphi_2(x) \cdot \left\{ \int_{x_0}^{x} \left[\sum_{j=1}^{M} |(D^{\nu}_{x_0} f_j)(w)|^p \right] dw \right\}^{\left(\frac{\lambda_\nu+\lambda_\beta}{p}\right)}, \qquad (55)
\end{aligned}
$$

all $x_0 \leq x \leq b$.

In (55), $\left(A_0(x)\big|_{\lambda_\alpha=0}\right)$ of $\varphi_2(x)$ is given in [6], Corollary 3, by equation (59).

Corollary 4 (to Theorem 3, $\lambda_\alpha = 0$, $p(t) = q(t) = 1$, $\lambda_\beta = \lambda_\nu = 1$, $p = 2$). *In detail*:

Let $\nu, \gamma_2 \geq 1$ *such that* $\nu - \gamma_2 \geq 1$ *and* $f_j \in C^{\nu}_{x_0}([a,b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here* $x, x_0 \in [a,b]$: $x \geq x_0$. *Then it holds*

$$
\begin{aligned}
&\int_{x_0}^{x} \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[|(D^{\gamma_2}_{x_0} f_{j+1})(w)| \, |(D^{\nu}_{x_0} f_j)(w)| \\
&\quad + |(D^{\gamma_2}_{x_0} f_j)(w)| \, |(D^{\nu}_{x_0} f_{j+1})(w)| \Big] \Bigg\} + \Big[|(D^{\gamma_2}_{x_0} f_M)(w)| \, |(D^{\nu}_{x_0} f_1)(w)| \\
&\quad + |(D^{\gamma_2}_{x_0} f_1)(w)| \, |(D^{\nu}_{x_0} f_M)(w)| \Big] \Bigg\} dw \\
&\leq \left(\frac{\sqrt{2}(x-x_0)^{(\nu-\gamma_2)}}{\Gamma(\nu-\gamma_2)\sqrt{\nu-\gamma_2}\sqrt{2\nu-2\gamma_2-1}} \right) \\
&\qquad \left\{ \int_{x_0}^{x} \left[\sum_{j=1}^{M} ((D^{\nu}_{x_0} f_j)(w))^2 \right] dw \right\}, \qquad (56)
\end{aligned}
$$

all $x_0 \leq x \leq b$.

Proof. From our Corollary 3 and Corollary 3 of [6], especially equation (59) there. ■

Corollary 5 (to Theorem 4, $\lambda_\alpha = \lambda_\beta = \lambda_\nu = 1$, $p = 3$, $p(t) = q(t) = 1$). *It holds*

$$
\begin{aligned}
&\int_{x_0}^{x} \Bigg[\sum_{j=1}^{M-1} \Big[|(D^{\gamma_1}_{x_0} f_j)(w)| \, |(D^{\gamma_2}_{x_0} f_{j+1})(w)| |(D^{\nu}_{x_0} f_j)(w)| \\
&\quad + |(D^{\gamma_2}_{x_0} f_j)(w)| \, |(D^{\gamma_1}_{x_0} f_{j+1})(w)| \, |(D^{\nu}_{x_0} f_{j+1})(w)| \Big] \\
&\quad + \Big[|(D^{\gamma_1}_{x_0} f_1)(w)| \, |(D^{\gamma_2}_{x_0} f_M)(w)| \, |(D^{\nu}_{x_0} f_1)(w)|
\end{aligned}
$$

$$+ \left| (D_{x_0}^{\gamma_2} f_1)(w) \right| \left| (D_{x_0}^{\gamma_1} f_M)(w) \right| \left| (D_{x_0}^{\nu} f_M)(w) \right| \Bigg] dw$$

$$\leq 2\varphi_3^*(x) \cdot \left[\int_{x_0}^{x} \left[\sum_{j=1}^{M} \left| (D_{x_0}^{\nu} f_j)(w) \right|^3 dw \right] \right], \tag{57}$$

all $x_0 \leq x \leq b$.

Here

$$\varphi_3^*(x) := \left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{6}} \right) A_0(x), \tag{58}$$

where in this special case

$$A_0(x) := \frac{4(x - x_0)^{(2\nu - \gamma_1 - \gamma_2)}}{\Gamma(\nu - \gamma_1)\Gamma(\nu - \gamma_2)[3(3\nu - 3\gamma_1 - 1)(3\nu - 3\gamma_2 - 1)(2\nu - \gamma_1 - \gamma_2)]^{2/3}}. \tag{59}$$

Proof. From Theorem 4 and equation (62) of [6], which is here equation (59). ■

Corollary 6 (to Theorem 5, $\lambda_\alpha = 1$, $\lambda_{\alpha+1} = \frac{1}{2}$, $p = \frac{3}{2}$, $p(t) = q(t) = 1$). *In detail*:

Let $\nu \geq 3$ *and* $\gamma_1 \geq 1$ *such that* $\nu - \gamma_1 \geq 2$. *Let* $f_j \in C_{x_0}^{\nu}([a, b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here* $x, x_0 \in [a, b]$: $x \geq x_0$. *Set*

$$\Phi^*(x) := \left(\sqrt{\frac{2}{3\nu - 3\gamma_1 - 2}} \right) \frac{(x - x_0)^{\left(\frac{3\nu - 3\gamma_1 - 1}{2}\right)}}{(\Gamma(\nu - \gamma_1))^{3/2}}, \tag{60}$$

all $x_0 \leq x \leq b$.

Then it holds

$$\int_{x_0}^{x} \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \Big[\left| (D_{x_0}^{\gamma_1} f_j)(w) \right| \sqrt{\left| (D_{x_0}^{\gamma_1 + 1} f_{j+1})(w) \right|}$$

$$+ \left| (D_{x_0}^{\gamma_1} f_{j+1})(w) \right| \sqrt{\left| (D_{x_0}^{\gamma_1 + 1} f_j)(w) \right|} \Big] \Bigg\}$$

$$+ \Big[\left| (D_{x_0}^{\gamma_1} f_1)(w) \right| \sqrt{\left| (D_{x_0}^{\gamma_1 + 1} f_M)(w) \right|}$$

$$+ \left| (D_{x_0}^{\gamma_1} f_M)(w) \right| \sqrt{\left| (D_{x_0}^{\gamma_1 + 1} f_1)(w) \right|} \Big] \Bigg\} dw$$

$$\leq 2\Phi^*(x) \cdot \left[\int_{x_0}^{x} \left(\sum_{j=1}^{M} \left| (D_{x_0}^{\nu} f_j)(w) \right|^{3/2} \right) dw \right], \tag{61}$$

all $x_0 \leq x \leq b$.

Proof. Based on Theorem 5 here, and equation (64) of [6] to establish coefficient $\Phi^*(x)$ in (61). ■

Corollary 7 (to Theorem 6, $p = 2(\lambda_\alpha + \lambda_\nu) > 1$, $p(t) = q(t) = 1$). *It holds*

$$\begin{aligned}
&\int_{x_0}^{x} \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_2} f_{j+1})(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\nu} \\
&\quad + |(D_{x_0}^{\gamma_2} f_j)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_{j+1})(w)|^{\lambda_\nu} \big] \Bigg\} \\
&\quad + \big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_{x_0}^{\gamma_2} f_M)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\nu} f_1)(w)|^{\lambda_\nu} \\
&\quad + |(D_{x_0}^{\gamma_2} f_1)(w)|^{\lambda_\alpha+\lambda_\nu} |(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha} |(D_{x_0}^{\nu} f_M)(w)|^{\lambda_\nu} \big] \Bigg\} dw \\
&\leq 2\tilde{T}(x) \left[\int_{x_0}^{x} \left(\sum_{j=1}^{M} |(D_{x_0}^{\nu} f_j)(w)|^{2(\lambda_\alpha+\lambda_\nu)} \right) dw \right], \qquad (62)
\end{aligned}$$

all $x_0 \leq x \leq b$.

Here $\tilde{T}(x)$ in (62) is given precisely by equations (66)–(70) of [6].

Corollary 8 (to Theorem 6, $p = 4$, $\lambda_\alpha = \lambda_\nu = 1$, $p(t) = q(t) = 1$). *It holds*

$$\begin{aligned}
&\int_{x_0}^{x} \Bigg\{ \Bigg\{ \sum_{j=1}^{M-1} \big[|(D_{x_0}^{\gamma_1} f_j)(w)| ((D_{x_0}^{\gamma_2} f_{j+1})(w))^2 |(D_{x_0}^{\nu} f_j)(w)| \\
&\quad + ((D_{x_0}^{\gamma_2} f_j)(w))^2 |(D_{x_0}^{\gamma_1} f_{j+1})(w)| \, |(D_{x_0}^{\nu} f_{j+1})(w)| \big] \Bigg\} \\
&\quad + \big[|(D_{x_0}^{\gamma_1} f_1)(w)| ((D_{x_0}^{\gamma_2} f_M)(w))^2 |(D_{x_0}^{\nu} f_1)(w)| \\
&\quad + ((D_{x_0}^{\gamma_2} f_1)(w))^2 |(D_{x_0}^{\gamma_1} f_M)(w)| \, |(D_{x_0}^{\nu} f_M)(w)| \big] \Bigg\} dw \\
&\leq 2\tilde{T}(x) \left[\int_{x_0}^{x} \left(\sum_{j=1}^{M} ((D_{x_0}^{\nu} f_j)(w))^4 \right) dw \right], \qquad (63)
\end{aligned}$$

all $x_0 \leq x \leq b$.

Here in (63) we have that $\tilde{T}(x) = T^*(x)$ of Corollary 8 in [6], for it see there equations (72)–(76).

Next we present the supremum case

Theorem 7. *Let* $\nu, \gamma_1, \gamma_2 \geq 1$ *such that* $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$ *and* $f_j \in C_{x_0}^{\nu}([a,b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here* $x, x_0 \in [a,b]: x \geq x_0$. *Consider* $p(x) \geq 0$ *continuous function on* $[x_0, b]$. *Let* $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. *Set*

$$\rho(x) := \frac{(x - x_0)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)} \|p(x)\|_\infty}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \qquad (64)$$

Then it holds

$$\int_{x_0}^{x} p(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu}$$
$$+|(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_{j+1}(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\Big]\Bigg\}$$
$$+\Big[|(D_{x_0}^{\gamma_1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu}f_1)(w)|^{\lambda_\nu}$$
$$+|(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\gamma_1}f_M)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\Big]\Bigg\}dw$$
$$\leq \rho(x)\Bigg\{\sum_{j=1}^{M}\{\|(D_{x_0}^{\nu}f_j)\|_\infty^{2(\lambda_\alpha+\lambda_\nu)}+\|(D_{x_0}^{\nu}f_j)\|_\infty^{2\lambda_\beta}\}\Bigg\}, \tag{65}$$

all $x_0 \leq x \leq b$.

Proof. Based on Theorem 7 of [6]. ■

Similarly we give

Theorem 8 (as in Theorem 7, $\lambda_\beta = 0$). *It holds*

$$\int_{x_0}^{x} p(w)\left(\sum_{j=1}^{M}|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu}\right)dw$$
$$\leq \left(\frac{(x-x_0)^{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+1)}\|p(x)\|_\infty}{(\nu\lambda_\alpha-\gamma_1\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}}\right)\cdot\left(\sum_{j=1}^{M}\|D_{x_0}^{\nu}f_j\|_\infty^{\lambda_\alpha+\lambda_\nu}\right), \tag{66}$$

all $x_0 \leq x \leq b$.

Proof. Based on Theorem 8 of [6]. ■

It follows

Theorem 9 (as in Theorem 7, $\lambda_\beta = \lambda_\alpha + \lambda_\nu$). *It holds*

$$\int_{x_0}^{x} p(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_{j+1})(w)|^{\lambda_\alpha+\lambda_\nu}|(D_{x_0}^{\nu}f_j)(w)|^{\lambda_\nu}$$
$$+|(D_{x_0}^{\gamma_2}f_j)(w)|^{\lambda_\alpha+\lambda_\nu}|(D_{x_0}^{\gamma_1}f_{j+1})(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_{j+1})(w)|^{\lambda_\nu}\Big]\Bigg\}$$
$$+\Big[|(D_{x_0}^{\gamma_1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2}f_M)(w)|^{\lambda_\alpha+\lambda_\nu}|(D_{x_0}^{\nu}f_1)(2)|^{\lambda_\nu}$$
$$+|(D_{x_0}^{\gamma_2}f_1)(w)|^{\lambda_\alpha+\lambda_\nu}|(D_{x_0}^{\gamma_1}f_M)(w)|^{\lambda_\alpha}|(D_{x_0}^{\nu}f_M)(w)|^{\lambda_\nu}\Big]\Bigg\}dw$$
$$\leq \left(\frac{2(x-x_0)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)}\|p(x)\|_\infty}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha+\nu\lambda_\nu-\gamma_2\lambda_\alpha-\gamma_2\lambda_\nu+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{(\lambda_\alpha+\lambda_\nu)}}\right)$$
$$\cdot\left(\sum_{j=1}^{M}\|D_{x_0}^{\nu}f_j\|_\infty^{2(\lambda_\alpha+\lambda_\nu)}\right), \tag{67}$$

all $x_0 \le x \le b$.

Proof. By Theorem 9 of [6]. ■

We continue with

Theorem 10 (as in Theorem 7, $\lambda_\nu = 0$, $\lambda_\alpha = \lambda_\beta$). *It holds*

$$\begin{aligned}
&\int_{x_0}^{x} p(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_1} f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2} f_{j+1})(w)|^{\lambda_\alpha} \\
&\quad + |(D_{x_0}^{\gamma_2} f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1} f_{j+1})(w)|^{\lambda_\alpha}\Big]\Bigg\} \\
&\quad + \Big[|(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_2} f_M)(w)|^{\lambda_\alpha} \\
&\quad + |(D_{x_0}^{\gamma_2} f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1} f_M)(w)|^{\lambda_\alpha}\Big]\Bigg\} dw \\
&\le\ 2\rho^*(x)\left[\sum_{j=1}^{M}\|D_{x_0}^{\nu} f_j\|_\infty^{2\lambda_\alpha}\right], \qquad (68)
\end{aligned}$$

all $x_0 \le x \le b$.

Here we have

$$\rho^*(x) := \left(\frac{(x-x_0)^{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)}\|p(x)\|_\infty}{(2\nu\lambda_\alpha-\gamma_1\lambda_\alpha-\gamma_2\lambda_\alpha+1)(\Gamma(\nu-\gamma_1+1))^{\lambda_\alpha}(\Gamma(\nu-\gamma_2+1))^{\lambda_\alpha}}\right). \qquad (69)$$

Proof. Based on Theorem 10 of [6]. ■

Next we give

Theorem 11 (as in Theorem 7, $\lambda_\alpha = 0$, $\lambda_\beta = \lambda_\nu$). *It holds*

$$\begin{aligned}
&\int_{x_0}^{x} p(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_2} f_{j+1})(w)|^{\lambda_\beta}|(D_{x_0}^{\nu} f_j)(w)|^{\lambda_\beta} \\
&\quad + |(D_{x_0}^{\gamma_2} f_j)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu} f_{j+1})(w)|^{\lambda_\beta}\Big]\Bigg\} \\
&\quad + \Big[|(D_{x_0}^{\gamma_2} f_M)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu} f_1)(w)|^{\lambda_\beta} \\
&\quad + |(D_{x_0}^{\gamma_2} f_1)(w)|^{\lambda_\beta}|(D_{x_0}^{\nu} f_M)(w)|^{\lambda_\beta}\Big]\Bigg\} dw \\
&\le\ 2\cdot\left(\frac{(x-x_0)^{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)}\|p(x)\|_\infty}{(\nu\lambda_\beta-\gamma_2\lambda_\beta+1)(\Gamma(\nu-\gamma_2+1))^{\lambda_\beta}}\right)\left[\sum_{j=1}^{M}\|D_{x_0}^{\nu} f_j\|_\infty^{2\lambda_\beta}\right], \qquad (70)
\end{aligned}$$

all $x_0 \le x \le b$.

Proof. Based on Theorem 11 of [6]. ■

Some special cases follow.

Corollary 9 (to Theorem 10, all as in Theorem 7, $\lambda_\nu = 0$, $\lambda_\alpha = \lambda_\beta$, $\gamma_2 = \gamma_1 + 1$). *It holds*

$$\begin{aligned}
&\int_{x_0}^{x} p(w)\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1+1}f_{j+1})(w)|^{\lambda_\alpha} \\
&\quad + |(D_{x_0}^{\gamma_1+1}f_j)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1}f_{j+1})(w)|^{\lambda_\alpha}\Big]\Bigg\} \\
&\quad + \Big[|(D_{x_0}^{\gamma_1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1+1}f_M)(w)|^{\lambda_\alpha} \\
&\quad + |(D_{x_0}^{\gamma_1+1}f_1)(w)|^{\lambda_\alpha}|(D_{x_0}^{\gamma_1}f_M)(w)|^{\lambda_\alpha}\Big]\Bigg\}dw \\
&\leq 2\cdot\left(\frac{(x-x_0)^{(2\nu\lambda_\alpha-2\gamma_1\lambda_\alpha-\lambda_\alpha+1)}\|p(x)\|_\infty}{(2\nu\lambda_\alpha-2\gamma_1\lambda_\alpha-\lambda_\alpha+1)(\nu-\gamma_1)^{\lambda_\alpha}(\Gamma(\nu-\gamma_1))^{2\lambda_\alpha}}\right) \\
&\quad\cdot\left[\sum_{j=1}^{M}\|D_{x_0}^{\nu}f_j\|_\infty^{2\lambda_\alpha}\right], \qquad (71)
\end{aligned}$$

all $x_0 \leq x \leq b$.

Proof. Based on Corollary 9 of [6]. ■

Corollary 10 (to Corollary 9). *In detail*:

Let $\nu, \gamma_1 \geq 1$ *such that* $\nu - \gamma_1 \geq 2$ *and* $f_j \in C_{x_0}^{\nu}([a,b])$ *with* $f_j^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n := [\nu]$, $j = 1, \ldots, M \in \mathbb{N}$. *Here* $x, x_0 \in [a,b]$: $x \geq x_0$. *Then*

$$\begin{aligned}
&\int_{x_0}^{x}\Bigg\{\Bigg\{\sum_{j=1}^{M-1}\Big[|(D_{x_0}^{\gamma_1}f_j)(w)|\,|(D_{x_0}^{\gamma_1+1}f_{j+1})(w)| \\
&\quad + |(D_{x_0}^{\gamma_1+1}f_j)(w)|\,|(D_{x_0}^{\gamma_1}f_{j+1})(w)|\Big]\Bigg\} \\
&\quad + \Big[|(D_{x_0}^{\gamma_1}f_1)(w)|\,|(D_{x_0}^{\gamma_1+1}f_M)(w)| \\
&\quad + |(D_{x_0}^{\gamma_1+1}f_1)(w)|\,|(D_{x_0}^{\gamma_1}f_M)(w)|\Big]\Bigg\}dw \\
&\leq \left(\frac{(x-x_0)^{2(\nu-\gamma_1)}}{(\nu-\gamma_1)^2(\Gamma(\nu-\gamma_1))^2}\right)\left(\sum_{j=1}^{M}\|D_{x_0}^{\nu}f_j\|_\infty^2\right), \qquad (72)
\end{aligned}$$

all $x_0 \leq x \leq b$.

Proof. Based on Corollary 10 of [6]. ■

Corollary 11 (to Corollary 10). *It holds*

$$\int_{x_0}^{x}\left(\sum_{j=1}^{M}|(D_{x_0}^{\gamma_1}f_j)(w)|\,|(D_{x_0}^{\gamma_1+1}f_j)(w)|\right)dw$$

$$\leq \left(\frac{(x-x_0)^{2(\nu-\gamma_1)}}{2(\nu-\gamma_1)^2(\Gamma(\nu-\gamma_1))^2}\right)\left(\sum_{j=1}^{M}\|D_{x_0}^{\nu}f_j\|_{\infty}^2\right), \tag{73}$$

all $x_0 \leq x \leq b$.

Proof. Based on equation (97) of [6]. ■

3 Applications

We present our first application.

Theorem 12. *Let* $\nu, \gamma_i \geq 1$, $\nu - \gamma_i \geq 1$, $i = 1, \ldots, r \in \mathbb{N}$, $n := [\nu]$, $f_j \in C_a^{\nu}([a,b])$, $j = 1, 2, 3, \ldots, M$, $f_j^{(i)}(a) = a_{ij} \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$. *Furthermore we have for* $j = 1, 2, \ldots, M$ *that*

$$(D_a^{\nu}f_j)(t) = F_j\big(t, \big\{(D_a^{\gamma_i}f_1)(t)\big\}_{i=1}^{r}, \big\{(D_a^{\gamma_i}f_2)(t)\big\}_{i=1}^{r}, \ldots, \big\{(D_a^{\gamma_i}f_M)(t)\big\}_{i=1}^{r}\big), \tag{74}$$

all $t \in [a,b]$.

Here F_j *are continuous functions on* $[a,b] \times (\mathbb{R}^r)^M$ *and satisfy the Lipschitz condition*

$$\begin{aligned}&\big|F_j(t, x_{11}, x_{12}, \ldots, x_{1r}, x_{21}, \ldots, x_{2r}, x_{31}, \ldots, x_{3r}, \ldots, x_{M1}, \ldots, x_{Mr})\\ &\quad - F_j(t, x'_{11}, x'_{12}, \ldots, x'_{1r}, x'_{21}, \ldots, x'_{2r}, x'_{31}, \ldots, x'_{3r}, x'_{M1}, \ldots, x'_{Mr})\big|\\ &\leq \sum_{i=1}^{r}\left(\sum_{\ell=1}^{M} q_{\ell,i,j}(t)|x_{\ell i} - x'_{\ell i}|\right),\end{aligned} \tag{75}$$

$j = 1, 2, \ldots, M$, *where all* $q_{\ell,i,j} \geq 0$, $1 \leq i \leq r$, *are continuous functions over* $[a,b]$.

Call

$$W := \max\big\{\|q_{\ell,i,j}\|_{\infty}, \ell, j = 1, 2, \ldots, M, i = 1, \ldots, r\big\}. \tag{76}$$

Assume here that

$$\phi^*(b) := W\left(\frac{1}{2} + \frac{M-1}{\sqrt{2}}\right)\left(\sum_{i=1}^{r}\left(\frac{(b-a)^{\nu-\gamma_i}}{\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}}\right)\right) < 1. \tag{77}$$

Then if system (74) has two M*-tuples of solutions* $(f_1, f_2, \ldots, f_M)$ *and* $(f_1^*, f_2^*, \ldots, f_M^*)$ *we prove that*

$$f_j = f_j^*, \quad j = 1, 2, \ldots, M,$$

that is we have uniqueness of solution.

Proof. Assume that there are two M-tuples of solutions $(f_1, f_2, \ldots, f_M)$ and $(f_1^*, \ldots, f_M^*)$ satisfying the system (74). Set $g_j := f_j - f_j^*$, $j = 1, 2, \ldots, M$.

Then $g_j^{(i)} = f_j^{(i)} - f_j^{*(i)}$ and $g_j^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$; $j = 1, 2, \ldots, M$. It holds

$$(D_a^\nu g_j)(t) = F_j\big(t, \big\{(D_a^{\gamma_i} f_1)(t)\big\}_{i=1}^r, \ldots, \big\{(D_a^{\gamma_i} f_M)(t)\big\}_{i=1}^r\big) - F_j\big(t, \big\{(D_a^{\gamma_i} f_1^*)(t)\big\}_{i=1}^r, \ldots, \big\{(D_a^{\gamma_i} f_M^*)(t)\big\}_{i=1}^r\big).$$

Therefore by (75) we get

$$|(D_a^\nu g_j)(t)| \le \sum_{i=1}^r \big[q_{1,i,j}(t)|(D_a^{\gamma_i} g_1)(t)| + q_{2,i,j}(t)|(D_a^{\gamma_i} g_2)(t)| + \cdots + q_{M,i,j}(t)|(D_a^{\gamma_i} g_M)(t)|\big].$$

And thus

$$|(D_a^\nu g_j)(t)| \le \sum_{i=1}^r \big[\|q_{1,i,j}\|_\infty |(D_a^{\gamma_i} g_1)(t)| + \|q_{2,i,j}\|_\infty |(D_a^{\gamma_i} g_2)(t)| + \cdots + \|q_{M,i,j}\|_\infty |(D_a^{\gamma_i} g_M)(t)|\big],$$

furthermore we have

$$|(D_a^\nu g_j)(t)| \le W\bigg\{\sum_{i=1}^r \big[|(D_a^{\gamma_i} g_1)(t)| + |(D_a^{\gamma_i} g_2)(t)| + \cdots + |(D_a^{\gamma_i} g_M)(t)|\big]\bigg\}. \tag{78}$$

Clearly (78) implies

$$\sum_{j=1}^M \big((D_a^\nu g_j)(t)\big)^2 \le W\bigg\{\sum_{i=1}^r \sum_{j=1}^M \big[|(D_a^{\gamma_i} g_1)(t)|\,|(D_a^\nu g_j)(t)| + |(D_a^{\gamma_i} g_2)(t)|\,|(D_a^\nu g_j)(t)| + \cdots + |(D_a^{\gamma_i} g_M)(t)|\,|(D_a^\nu g_j)(t)|\big]\bigg\}, \tag{79}$$

$j = 1, 2, \ldots, M$.

Integrating (79) we observe

$$I := \int_a^b \left(\sum_{j=1}^M \big((D_a^\nu g_j)(t)\big)^2\right) dt \le W\bigg\{\sum_{i=1}^r \sum_{j=1}^M \bigg[\int_a^b |(D_a^{\gamma_i} g_1)(t)|\,|(D_a^\nu g_j)(t)|\,dt + \int_a^b |(D_a^{\gamma_i} g_2)(t)|\,|(D_a^\nu g_j)(t)|\,dt + \cdots + \int_a^b |(D_a^{\gamma_i} g_M)(t)|\,|(D_a^\nu g_j)(t)|\,dt\bigg]\bigg\}. \tag{80}$$

That is

$$I \leq W\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg(\int_a^b\Bigg(\sum_{\lambda=1}^{M}|(D_a^{\gamma_i}g_\lambda)(t)|\,|(D_a^{\nu}g_\lambda)(t)|\Bigg)dt\Bigg) + \sum_{\substack{\tau,m\in\{1,\ldots,M\}\\ \tau\neq m}}\Bigg(\int_a^b(|(D_a^{\gamma_i}g_m)(t)|\,|(D_a^{\nu}g_\tau)(t)| + |(D_a^{\gamma_i}g_\tau)(t)|\,|(D_a^{\nu}g_m)(t)|\Bigg)\,dt\Bigg)\Bigg]\Bigg\}. \tag{81}$$

Using Corollary 2 from here and Corollary 4 of [6] we obtain

$$I \leq W\Bigg\{\sum_{i=1}^{r}\Bigg[\Bigg(\frac{(b-a)^{\nu-\gamma_i}}{2\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}}\Bigg)I + \Bigg(\frac{(b-a)^{\nu-\gamma_i}}{\sqrt{2}\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}}\Bigg)(M-1)I\Bigg]\Bigg\}. \tag{82}$$

I.e. we got that

$$I \leq \phi^*(b)\cdot I. \tag{83}$$

If $I \neq 0$ then $\phi^*(b) \geq 1$, a contradiction by the assumption that $\phi^*(b) < 1$, see (77). Therefore $I = 0$, implying that

$$\sum_{\lambda=1}^{M}\big((D_a^{\nu}g_\lambda)(t)\big)^2 = 0, \quad \text{a.e. in } [a,b].$$

I.e.

$$(D_a^{\nu}g_\lambda)^2(t) = 0, \quad \text{a.e. in } [a,b].$$

That is

$$(D_a^{\nu}g_\lambda)(t) = 0, \quad \lambda = 1,2,\ldots,M, \text{ a.e. in } [a,b].$$

But for $\lambda = 1,2,\ldots,M$ we got that

$$g_\lambda^{(i)}(a) = 0, \quad 0 \leq i \leq n-1.$$

Hence from fractional Taylor's Theorem 1 we get that $g_\lambda(t) = 0$ on $[a,b]$. That is

$$f_\lambda = f_\lambda^*, \quad \lambda = 1,2,\ldots,M,$$

proving the uniqueness argument of this theorem. ■

It follows another related application.

Theorem 13. *Let $\nu,\gamma_i \geq 1$, $\nu-\gamma_i \geq 1$, $i = 1,\ldots,r \in \mathbb{N}$, $n := [\nu]$, $f_j \in C_a^{\nu}([a,b])$, $j = 1,2,\ldots,M$; $f_j^{(i)}(a) = 0$, $i = 0,1,\ldots,n-1$, and $(D_a^{\nu}f_j)(a) =$*

$A_j \in \mathbb{R}$. *Furthermore for* $a \le t \le b$ *we have holding the system of fractional differential equations*

$$(D_a^\nu f_j)'(x) = F_j\big(t, (\{(D_a^{\gamma_i} f_\lambda)(t)\}_{i=1}^r, (D_a^\nu f_\lambda)(t)); \\ \lambda = 1, 2, \ldots, M\big), \quad j = 1, 2, \ldots, M. \tag{84}$$

Here F_j *are continuous functions on* $[a,b] \times (\mathbb{R}^{r+1})^M$ *such that*

$$|F_j(t, x_{11}, x_{12}, \ldots, x_{1r}, x_{1,r+1}; x_{21}, x_{22}, \ldots, x_{2r}, x_{2,r+1}; x_{31}, x_{32}, \ldots, x_{3,r+1}; \\ x_{M1}, x_{M2}, \ldots, x_{M,r+1})| \le \sum_{i=1}^r \left(\sum_{\ell=1}^M q_{\ell,i,j}(t) |x_{\ell i}| \right), \tag{85}$$

where

$$q_{\ell,i,j}(t) \ge 0, \quad 1 \le i \le r; \quad \ell, j = 1, 2, \ldots, M,$$

are continuous functions on $[a,b]$.

Call

$$W := \max\big\{ \|q_{\ell,i,j}\|_\infty; \ell, j = 1, 2, \ldots, M, i = 1, \ldots, r \big\}. \tag{86}$$

Also we set $(a \le x \le b)$

$$\theta(x) := \sum_{\lambda=1}^M \big((D_a^\nu f_\lambda)(x)\big)^2, \tag{87}$$

$$\rho := \sum_{\lambda=1}^M A_\lambda^2, \tag{88}$$

$$Q(x) := W\big(1 + \sqrt{2}(M-1)\big) \left(\sum_{i=1}^r \left(\frac{(x-a)^{\nu-\gamma_i}}{\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}} \right) \right) \tag{89}$$

and

$$\chi(x) := \sqrt{\rho} \cdot \left\{ 1 + Q(x) \cdot e^{\left(\int_a^b Q(s)ds\right)} \cdot \left[\int_a^x \left(e^{-\left(\int_a^t Q(s)ds\right)} \right) dt \right] \right\}^{1/2}. \tag{90}$$

Then it holds

$$\sqrt{\theta(x)} \le \chi(x), \quad a \le x \le b. \tag{91}$$

Consequently we get

$$|(D_a^\nu f_j)(x)| \le \chi(x), \tag{92}$$

$$|f_j(x)| \le \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \chi(t)\, dt, \tag{93}$$

all $a \le x \le b$, $j = 1, 2, \ldots, M$. *Also it holds*

$$|(D_a^{\gamma_i} f_j)(x)| \le \frac{1}{\Gamma(\nu-\gamma_i)} \int_a^x (x-t)^{\nu-\gamma_i-1} \chi(t)\, dt, \tag{94}$$

all $a \le x \le b$, $j = 1, 2, \ldots, M$, $i = 1, \ldots, r$.

Proof. We easily get that $(a \le x \le b)$

$$\int_a^x (D_a^\nu f_j)(t)(D_a^\nu f_j)'(t)dt = \int_a^x (D_a^\nu f_j)(t) \cdot F_j\big(t, (\{(D_a^{\gamma_i} f_\lambda)(t)\}_{i=1}^r, (D_a^\nu f_\lambda)(t));\ \lambda = 1, 2, \ldots, M\big)\, dt. \tag{95}$$

Hence we obtain

$$\begin{aligned}\frac{((D_a^\nu f_j)(t))^2}{2}\bigg|_a^x &\le \int_a^x |(D_a^\nu f_j)(t)|\,|F_j \cdots|dt \\ &\le \int_a^x |(D_a^\nu f_j)(t)|\bigg[\sum_{i=1}^r \bigg(\sum_{\ell=1}^M q_{\ell,i,j}(t)|(D_a^{\gamma_i} f_\ell)(t)|\bigg]\, dt \\ &\le \sum_{i=1}^r \bigg(\sum_{\ell=1}^M \|q_{\ell,i,j}\|_\infty \int_a^x |(D_a^\nu f_j)(t)|\,|(D_a^{\gamma_i} f_\ell)(t)|\, dt\bigg) \\ &\le W\bigg(\sum_{i=1}^r \sum_{\ell=1}^M \bigg(\int_a^x |(D_a^\nu f_j)(t)|\,|(D_a^{\gamma_i} f_\ell)(t)|\, dt\bigg)\bigg).\end{aligned}$$

Thus we have for $j = 1, \ldots, M$ that

$$\begin{aligned}\big((D_a^\nu f_j)(x)\big)^2 \le A_j^2 + 2W\bigg\{&\sum_{i=1}^r \sum_{\ell=1}^M \\ &\cdot\bigg(\int_a^x |(D_a^\nu f_j)(t)|\,|(D_a^{\gamma_i} f_\ell)(t)|\, dt\bigg)\bigg\}.\end{aligned} \tag{96}$$

Consequently it holds

$$\begin{aligned}\theta(x) &\le \rho + 2W\bigg\{\sum_{i=1}^r \bigg(\sum_{j=1}^M \sum_{\ell=1}^M \bigg(\int_a^x |(D_a^\nu f_j)(t)|\,|(D_a^{\gamma_i} f_\ell)(t)|\, dt\bigg)\bigg)\bigg\} \\ &= \rho + 2W\bigg\{\sum_{i=1}^r \bigg\{\int_a^x \bigg(\sum_{\lambda=1}^M |(D_a^{\gamma_i} f_\lambda)(t)|\,|(D_a^\nu f_\lambda)(t)|\bigg)\, dt \\ &\quad + \sum_{\substack{\tau,m\in\{1,\ldots,M\}\\ \tau\ne m}} \bigg(\int_a^x (|(D_a^{\gamma_i} f_m)(t)|\,|(D_a^\nu f_\tau)(t)| \\ &\quad + |(D_a^{\gamma_i} f_\tau)(t)|\,|(D_a^\nu f_m)(t)|)\, dt\bigg)\bigg\}\bigg\}.\end{aligned} \tag{97}$$

Using Corollary 2 from here and Corollary 4 of [6] we obtain

$$\begin{aligned}\theta(x) \le \rho + 2W\bigg\{\sum_{i=1}^r \bigg\{&\bigg(\frac{(x-a)^{\nu-\gamma_i}}{2\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}}\bigg)\bigg(\int_a^x \theta(t)\, dt\bigg) \\ &+ \bigg(\frac{(x-a)^{\nu-\gamma_i}}{\sqrt{2}\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}\sqrt{2\nu-2\gamma_i-1}}\bigg)(M-1)\bigg(\int_a^x \theta(t)\, dt\bigg)\bigg\}\bigg\}.\end{aligned} \tag{98}$$

Hence we have

$$\theta(x) \leq \rho + Q(x) \int_a^x \theta(t)\, dt, \quad \text{all } a \leq x \leq b. \tag{99}$$

Here $\rho \geq 0$, $Q(x) \geq 0$, $Q(a) = 0$, $\theta(x) \geq 0$, all $a \leq x \leq b$. As in the proof of Theorem 13 of [6], see also [5], we get (91) and (92). Using (9) we get (93), and using (10) we establish (94). ■

Finally we give a specialized application.

Theorem 14. *Let $a \neq b$, $\nu \geq 3$, $\gamma_i \geq 1$, $\nu - \gamma_i \geq 1$, $i = 1, \ldots, r \in \mathbb{N}$, $n := [\nu]$, $f_j \in C_a^{\nu}([a,b])$, $j = 1, 2, \ldots, M$; $f_j^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$, and*

$$(D_a^{\nu} f_j)(a) = A_j \in \mathbb{R}. \tag{100}$$

Furthermore for $a \leq t \leq b$ we have holding the system of fractional differential equations

$$(D_a^{\nu} f_j)'(t) = F_j\big(t, (\{(D_a^{\gamma_i} f_\ell)(t)\}_{i=1}^r, (D_a^{\nu} f_\ell)(t)); \\ \ell = 1, \ldots, M\big), \quad \text{for } j = 1, 2, \ldots, M. \tag{101}$$

For fixed $i_* \in \{1, \ldots, r\}$ we assume that $\gamma_{i_*+1} = \gamma_{i_*} + 1$, and $\nu - \gamma_{i_*} \geq 2$, where γ_{i_*}, $\gamma_{i_*+1} \in \{\gamma_1, \ldots, \gamma_r\}$. Call $k := \gamma_{i_*}$, $\gamma := \gamma_{i_*} + 1$, i.e. $\gamma = k + 1$.

Here F_j are continuous functions on $[a,b] \times (\mathbb{R}^{r+1})^M$ such that

$$\begin{aligned} &|F_j(t, x_{11}, x_{12}, \ldots, x_{1r}, x_{1,r+1}; x_{21}, x_{22}, \ldots, x_{2r}, x_{2,r+1}; \\ &\quad x_{31}, x_{32}, \ldots, x_{3r}, x_{3,r+1}; \ldots; x_{M1}, x_{M2}, \ldots, x_{Mr}, x_{M,r+1})| \\ &\leq \left\{ \left\{ \sum_{\ell=1}^{M-1} \left(q_{\ell,1,j}(t) |x_{\ell i_*}| \sqrt{|x_{\ell+1,i_*+1}|} + q_{\ell,2,j}(t) |x_{\ell+1,i_*}| \sqrt{|x_{\ell,i_*+1}|} \right) \right\} \right. \\ &\quad \left. + \left(q_{M,1,j}(t) |x_{1i_*}| \sqrt{|x_{M,i_*+1}|} + q_{M,2,j}(t) |x_{Mi_*}| \sqrt{|x_{1,i_*+1}|} \right) \right\}, \end{aligned} \tag{102}$$

where all $0 \leq q_{\ell,1,j}$, $q_{\ell,2,j} \not\equiv 0$ are continuous functions over $[a,b]$.

Put

$$W := \max\{ \|q_{\ell,1,j}\|_\infty, \|q_{\ell,2,j}\|_\infty \}_{\ell,j=1}^M. \tag{103}$$

Also set

$$\theta(x) := \sum_{j=1}^M |(D_a^{\nu} f_j)(x)|, \quad a \leq x \leq b, \tag{104}$$

$$\rho := \sum_{j=1}^M |A_j|, \tag{105}$$

$$\Phi^*(x) := \left(\sqrt{\frac{2}{3\nu - 3k - 2}} \right) \frac{(x-a)^{\left(\frac{3\nu-3k-1}{2}\right)}}{(\Gamma(\nu-k))^{3/2}}, \tag{106}$$

all $a \le x \le b$, *and*

$$Q(x) := 2MW\Phi^*(x), \quad a \le x \le b, \tag{107}$$

$$\sigma := \|Q(x)\|_\infty, \quad a \le x \le b. \tag{108}$$

We assume that

$$(b-a)\sigma\sqrt{\rho} < 2. \tag{109}$$

Call

$$\tilde{\varphi}(x) := \rho + Q(x) \cdot \left[\frac{4\rho^{3/2}(x-a) - \sigma\rho^2(x-a)^2}{(2-\sigma\sqrt{\rho}(x-a))^2}\right], \quad \textit{all } a \le x \le b. \tag{110}$$

Then it holds

$$\theta(x) \le \tilde{\varphi}(x), \quad \textit{all } a \le x \le b, \tag{111}$$

in particular we have

$$|(D_a^\nu f_j)(x)| \le \tilde{\varphi}(x), \quad j = 1, \ldots, M, \quad \textit{all } a \le x \le b. \tag{112}$$

Furthermore we get

$$|f_j(x)| \le \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \tilde{\varphi}(t)\, dt, \tag{113}$$

and

$$|(D_a^{\gamma_i} f_j)(x)| \le \frac{1}{\Gamma(\nu-\gamma_i)} \int_a^x (x-t)^{\nu-\gamma_i-1} \tilde{\varphi}(t)\, dt, \tag{114}$$

$j = 1, \ldots, M$; $i = 1, \ldots, r$; *all* $a \le x \le b$.

Proof. Notice that $W > 0$ and $\sigma > 0$. For $a \le x \le b$ we get

$$\int_a^x (D_a^\nu f_j)'(t)dt = \int_a^x F_j\big(t, (\{D_a^{\gamma_i} f_\ell)(t)\}_{i=1}^r, (D_a^\nu f_\ell)(t)); \\ \ell = 1, \ldots, M\big)\, dt, \quad j = 1, \ldots, M. \tag{115}$$

That is

$$(D_a^\nu f_j)(x) = A_j + \int_a^x F_j(t, \ldots)\, dt. \tag{116}$$

Then we observe

$$\begin{aligned} |(D_a^\nu f_j)(x)| &\le |A_j| + \int_a^x |F_j(t, \ldots)|\, dt \\ &\le |A_j| + \int_a^x \Bigg\{ \Bigg\{ \sum_{\ell=1}^{M-1} \big(q_{\ell,1,j}(t) |(D_a^{\gamma_{i_*}} f_\ell)(t)| \sqrt{|(D_a^{\gamma_{i_*}+1} f_{\ell+1})(t)|} \\ &\quad + q_{\ell,2,j}(t) |(D_a^{\gamma_{i_*}} f_{\ell+1})(t)| \sqrt{|(D_a^{\gamma_{i_*}+1} f_\ell)(t)|} \big) \Bigg\} \\ &\quad + \big(q_{M,1,j}(t) |(D_a^{\gamma_{i_*}} f_1)(t)| \sqrt{|(D_a^{\gamma_{i_*}+1} f_M)(t)|} \\ &\quad + q_{M,2,j}(t) |(D_a^{\gamma_{i_*}} f_M)(t)| \sqrt{|(D_a^{\gamma_{i_*}+1} f_1)(t)|} \big) \Bigg\} dt. \end{aligned} \tag{117}$$

Thus

$$|(D_a^\nu f_j)(x)| \le |A_j| + W\Bigg(\int_a^x \Bigg\{\Bigg\{\sum_{\ell=1}^{M-1}\Big(|(D_a^k f_\ell)(t)|\sqrt{|(D_a^{k+1}f_{\ell+1})(t)|}$$
$$+ |(D_a^k f_{\ell+1})(t)|\sqrt{|(D_a^{k+1}f_\ell)(t)|}\Big)\Bigg\} + \Big(|(D_a^k f_1)(t)|\sqrt{|(D_a^{k+1}f_M)(t)|}$$
$$+ |(D_a^k f_M)(t)|\sqrt{|(D_a^{k+1}f_1)(t))|}\Bigg\}\, dt. \tag{118}$$

By Corollary 6 we obtain

$$|(D_a^\nu f_j)(x)| \le |A_j| + 2W\Phi^*(x)\left(\int_a^x \left(\sum_{\ell=1}^{M}|(D_a^\nu f_\ell)(t)|^{3/2}\right) dt\right), \tag{119}$$

$j = 1, 2, \ldots, M$.

Therefore by adding all of inequalities (119) we get

$$\theta(x) \le \rho + 2M\Phi^*(x)W\left(\int_a^x \left(\sum_{\ell=1}^{M}|(D_a^\nu f_\ell)(t)|^{3/2}\right) dt\right)$$
$$\overset{\text{(by (11))}}{\le} \rho + 2M\Phi^*(x)W\left(\int_a^x \left(\sum_{\ell=1}^{M}|(D_a^\nu f_\ell)(t)|\right)^{3/2} dt\right). \tag{120}$$

I.e.

$$\theta(x) \le \rho + (2M\Phi^*(x)W)\left(\int_a^x (\theta(t))^{3/2}\, dt\right), \quad \text{all } a \le x \le b. \tag{121}$$

More precisely we get that

$$\theta(x) \le \rho + Q(x)\left(\int_a^x (\theta(t))^{3/2}\, dt\right), \quad a \le x \le b. \tag{122}$$

Notice that $\theta(x) \ge 0$, $\rho \ge 0$, $Q(x) \ge 0$ and $Q(a) = 0$ by $\Phi^*(a) = 0$. Acting here as in the proof of Theorem 14 of [6] we derive (111) and (112). Using (9) we get (113), and using (10) we establish (114). ■

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Moments of the Skew t Distribution

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Abstract

A random variable X is said to have the skew-t distribution if its pdf is $f(x) = 2g(x)G(\lambda x)$, where $g(\cdot)$ and $G(\cdot)$, respectively, denote the pdf and the cdf of the Student's t distribution with degrees of freedom ν. The moments of this distribution appear not to have been studied in detail. The only work that appears to give some details is Gupta *et al.* [*Random Operators and Stochastic Equations*, **10**, 2002, 133–140], where expressions for the first four moments are given. But these expressions appear to be incorrect. In this paper, we derive general expressions for the nth moment of X by considering the cases ν odd and ν even separately. These expressions turn out to involve sums of the Gauss hypergeometric function. We also provide closed form expressions for the moments of X for the particular cases $\nu = 2, \ldots, 10$.

1. Introduction

The Student's t distribution with degrees of freedom ν has the probability density function (pdf) specified by

$$g(x) = \frac{1}{\sqrt{\nu}B(\nu/2, 1/2)}\left(1+\frac{x^2}{\nu}\right)^{-(1+\nu)/2}, \tag{1.1}$$

where $-\infty < x < \infty$, $\nu > 0$ is an integer and $B(\cdot,\cdot)$ denotes the Beta function defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Nadarajah and Kotz (2003) have shown that the cumulative distribution function (cdf) corresponding to (1.1) can be expressed by:

$$G(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{\sqrt{\nu}}\right) + \frac{1}{2\pi}\sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right)\frac{\nu^{l-1/2}x}{\left(\nu+x^2\right)^l} \tag{1.2}$$

if ν is odd and by

$$G(x) = \frac{1}{2} + \frac{1}{2\pi}\sum_{l=1}^{\nu/2} B\left(l-\frac{1}{2},\frac{1}{2}\right)\frac{\nu^{l-1}x}{\left(\nu+x^2\right)^{l-1/2}} \tag{1.3}$$

if ν is even. A random variable X is said to have skew-t distribution if its pdf is

$$f(x) = 2g(x)G(\lambda x), \tag{1.4}$$

where $x \in \Re$ and $\lambda \in \Re$. The Student's t distribution given by (1.1) has major applications in the construction of tests and confidence intervals and in Bayesian analysis. It has also attracted interesting applications in the modeling of depth map data, prices of speculative assets such as stocks, and the phase derivative (random frequency of a narrowband mobile channel) of air components in an urban environment. The main feature of the skew-t distribution in (1.4) is that a new parameter λ is introduced to control skewness and kurtosis. Thus, (1.4) allows for a greater degree of flexibility and we can expect this to be useful in many more practical situations.

It follows from (1.4) that the pdf of X is

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{\nu}B\left(\nu/2,1/2\right)}\left(1+\dfrac{x^2}{\nu}\right)^{-(1+\nu)/2}\left\{1+\dfrac{2}{\pi}\arctan\left(\dfrac{\lambda x}{\sqrt{\nu}}\right)\right. \\ \qquad\qquad \left.+\dfrac{\lambda}{\pi}\displaystyle\sum_{l=1}^{(\nu-1)/2} B\left(l,\frac{1}{2}\right)\frac{\nu^{l-1/2}x}{\left(\nu+\lambda^2x^2\right)^{l}}\right\}, & \text{for } \nu \text{ odd}, \\ \dfrac{1}{\sqrt{\nu}B\left(\nu/2,1/2\right)}\left(1+\dfrac{x^2}{\nu}\right)^{-(1+\nu)/2}\left\{1\right. \\ \qquad\qquad \left.+\dfrac{\lambda}{\pi}\displaystyle\sum_{l=1}^{\nu/2} B\left(l-\frac{1}{2},\frac{1}{2}\right)\frac{\nu^{l-1}x}{\left(\nu+\lambda^2x^2\right)^{l-1/2}}\right\}, & \text{for } \nu \text{ even}, \end{cases} \tag{1.5}$$

where $x \in \Re$ and $\lambda \in \Re$. The particular case of (1.1) and (1.2)–(1.3) for $\nu = 1$ is the well known Cauchy distribution. Thus, when $\nu = 1$, (1.5) reduces to what is known as the skew-Cauchy distribution with the pdf

$$f(x) = \frac{1}{\pi\left(1+x^2\right)}\left\{1+\frac{2}{\pi}\arctan(\lambda x)\right\}.$$

When $\lambda = 0$, (1.5) reduces to the standard Student's t pdf (1.1). Figure 1 illustrates the shape of the pdf (1.5) for $\lambda = 0, 1, 2, 5, 10$ and $\nu = 1$.

The moments of the skew-t distribution appear not to have been studied in detail. The only work that appears to give some details is Gupta *et al.* (2002), where expressions for the first four moments are given. However, these expressions appear to be incorrect. In this paper, we derive general formulas for the nth moment of the distribution by considering the cases ν odd and ν even separately. We also provide closed form expressions for the particular cases $\nu = 2, \ldots, 10$.

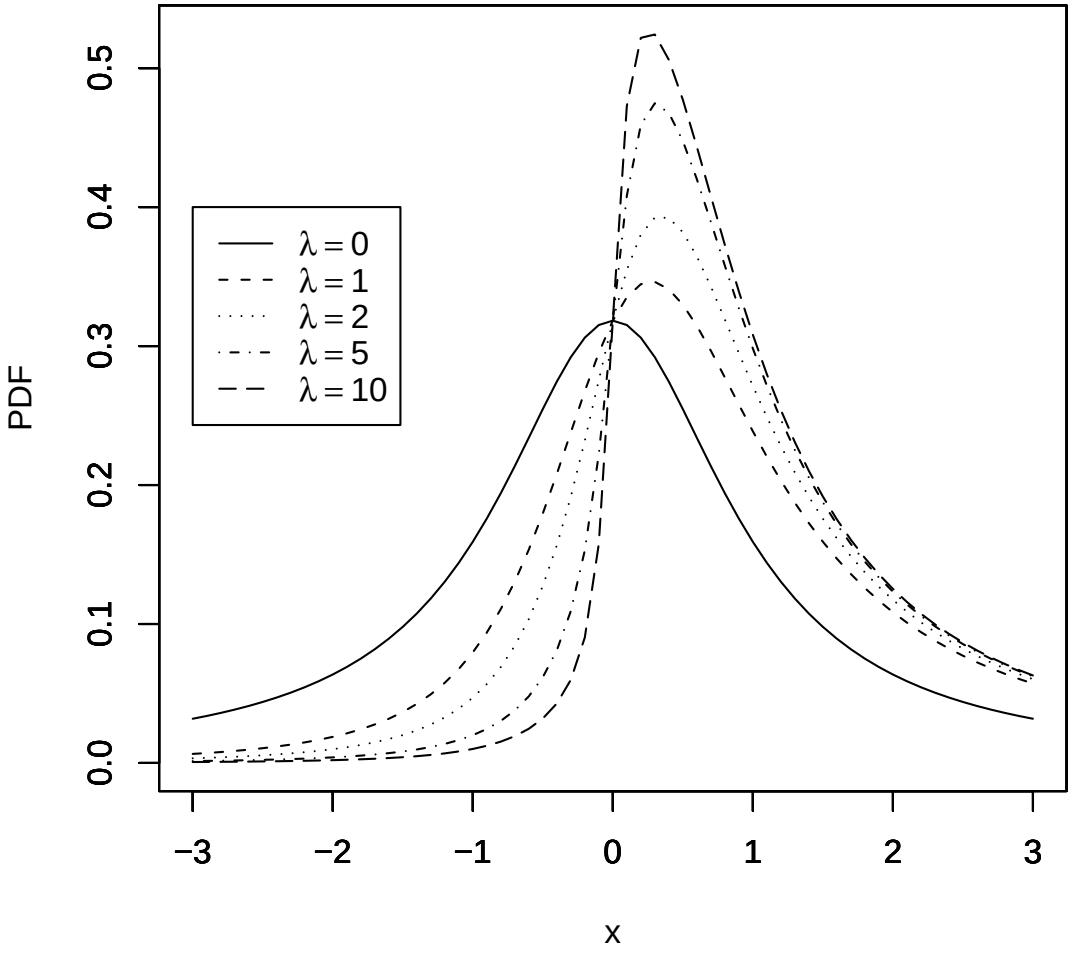


Figure 1. The skew-t pdf (1.5) for $\lambda = 0, 1, 2, 5, 10$ and $\nu = 1$.

In addition to those mentioned above, the calculations of the paper make use of the following special functions: the complete elliptical integral of the first kind defined by

$$\mathbf{K}(a) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}} dx;$$

the complete elliptical integral of the second kind defined by

$$\mathbf{E}(a) = \int_0^1 \frac{\sqrt{1-a^2x^2}}{\sqrt{1-x^2}} dx;$$

and, the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. The nth Moment

By Lemma 2 in Gupta *et al.* (2002), the even order moments of X are the same as those of the standard Student's t distribution given by (1.1), which are well known in the literature (see, for example, Chapter 28, Johnson *et al.*, 1995). Thus,

$$E\left(X^n\right) = \frac{\nu^{n/2} B\left(\frac{n+1}{2}, \frac{\nu-n}{2}\right)}{B\left(\frac{1}{2}, \frac{\nu}{2}\right)} \tag{2.1}$$

when $n < \nu$ is even. Theorem 1 derives the expression for the odd order moments of X for the case of odd ν.

Theorem 1 If X is a random variable having the pdf (1.5) for odd ν and $n < \nu$ is odd then

$$\begin{aligned} E\left(X^n\right) &= I_n + \frac{\lambda \nu^{n/2}}{\pi B\left(\nu/2, 1/2\right)} \sum_{k=1}^{(\nu-1)/2} B\left(k, \frac{1}{2}\right) B\left(\frac{n}{2}+1, k+\frac{\nu-n-1}{2}\right) \\ &\quad \times {}_2F_1\left(k, \frac{n}{2}+1; k+\frac{1+\nu}{2}; 1-\lambda^2\right), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} I_n &= \frac{2\nu^{(n-1)/2}}{\pi B\left(\nu/2, 1/2\right)} \sum_{k=0}^{\infty} \frac{(2k)!\lambda^{2k+1}}{4^k (k!)^2 (2k+1)} B\left(\frac{n}{2}+k+1, \frac{\nu-n}{2}\right) \\ &\quad \times {}_2F_1\left(k+\frac{1}{2}, \frac{n}{2}+k+1; \frac{\nu}{2}+k+1; 1-\lambda^2\right). \end{aligned}$$

Proof: Using (1.5), one can write

$$E\left(X^n\right) = \frac{2M_n}{\pi\sqrt{\nu} B\left(\nu/2, 1/2\right)} + \frac{\lambda}{\pi\sqrt{\nu} B\left(\nu/2, 1/2\right)} \sum_{k=1}^{(\nu-1)/2} \nu^{k-1/2} B\left(k, \frac{1}{2}\right) N_{n,k}, \tag{2.3}$$

where M_n and $N_{n,k}$ are the integrals

$$M_n = \int_{-\infty}^{\infty} x^n \left(1+\frac{x^2}{\nu}\right)^{-(1+\nu)/2} \arctan\left(\frac{\lambda x}{\sqrt{\nu}}\right) dx$$

and

$$N_{n,k} = \int_{-\infty}^{\infty} x^{n+1} \left(\nu+\lambda^2 x^2\right)^{-k} \left(1+\frac{x^2}{\nu}\right)^{-(1+\nu)/2} dx.$$

By expanding the arctan(·) term using equation (1.644.1) of Gradshteyn and Ryzhik (2000) and then integrating term by term, one can rewrite M_n as

$$M_n = \nu^{(1+\nu)/2} \sum_{k=0}^{\infty} \frac{(2k)! L_{n,k}}{4^k (k!)^2 (2k+1)}, \tag{2.4}$$

where

$$L_{n,k} = \int_0^{\infty} x^{n/2+k} \left(\frac{\nu}{\lambda^2} + x\right)^{-(k+1/2)} (\nu + x)^{-(1+\nu)/2} \, dx.$$

The integral $N_{n,k}$ can be simplified to

$$N_{n,k} = \frac{\nu^{(1+\nu)/2}}{\lambda^{2k}} \int_0^{\infty} x^{n/2} \left(\frac{\nu}{\lambda^2} + x\right)^{-k} (\nu + x)^{-(1+\nu)/2} \, dx.$$

Both $L_{n,k}$ and $N_{n,k}$ can be expressed in terms of the Gauss hypergeometric function by using equation (3.197.1) in Gradshteyn and Ryzhik (2000). It becomes that

$$L_{n,k} = \nu^{(n-\nu)/2} \lambda^{2k+1} B\left(k + 1 + \frac{n}{2}, \frac{\nu - n}{2}\right) \, {}_2F_1\left(k + \frac{1}{2}, k + 1 + \frac{n}{2}; k + 1 + \frac{\nu}{2}; 1 - \lambda^2\right) \tag{2.5}$$

and

$$N_{n,k} = \nu^{n/2+1-k} B\left(1 + \frac{n}{2}, k + \frac{\nu - n - 1}{2}\right) \, {}_2F_1\left(k, 1 + \frac{n}{2}; k + \frac{1+\nu}{2}; 1 - \lambda^2\right). \tag{2.6}$$

The result follows on substituting (2.5)–(2.6) into (2.3)–(2.4). ■

Theorem 2 is the analogue of Theorem 1 for the case of even ν. The proof is similar and is thus avoided.

Theorem 2 If X is a random variable having the pdf (1.5) for even ν and $n < \nu$ is odd then

$$E(X^n) = \frac{\lambda \nu^{n/2}}{\pi B(\nu/2, 1/2)} \sum_{k=1}^{\nu/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) B\left(\frac{n}{2} + 1, k - 1 + \frac{\nu - n}{2}\right) \times {}_2F_1\left(k - \frac{1}{2}, \frac{n}{2} + 1; k + \frac{\nu}{2}; 1 - \lambda^2\right). \tag{2.7}$$

3. Particular Cases

Here, we derive particular forms of (2.1), (2.2) and (2.7) for $\nu = 2, \ldots, 10$. In our calculations, we have used various special properties of the Gauss hypergeometric function (see, for example, Section 7.3 of Prudnikov *et al.* (1986, volume 3)). When ν is odd the expressions for $E(X^n)$ involve the infinite sum I_n which should be computed numerically. When ν is even the expressions involve the complete elliptical integrals of the first kind and second kind.

Corollary 1 If X is a random variable having the pdf (1.5) with $\nu = 2$ then

$$E(X) = \frac{\sqrt{2}}{\lambda^2 - 1}\left\{\lambda^2 \mathbf{E}(\mu) - \mathbf{K}(\mu)\right\},$$

where $\mu = \sqrt{(\lambda^2 - 1)/\lambda^2}$.

Corollary 2 If X is a random variable having the pdf (1.5) with $\nu = 3$ then

$$\begin{aligned} E(X) &= I_1 + \frac{\sqrt{3}\lambda}{\pi(1+\delta\lambda)^2}, \\ E\left(X^2\right) &= 3, \end{aligned}$$

where $\delta = \text{sign}(\lambda)$.

Corollary 3 If X is a random variable having the pdf (1.5) with $\nu = 4$ then

$$\begin{aligned} E(X) &= \frac{\lambda^2}{2\left(\lambda^2-1\right)^3}\left\{\left(2\lambda^4 - 7\lambda^2 - 3\right)\mathbf{E}(\mu) + \left(9 - \lambda^2\right)\mathbf{K}(\mu)\right\}, \\ E\left(X^2\right) &= 2, \\ E\left(X^3\right) &= \frac{2}{\left(\lambda^2-1\right)^3}\left\{\lambda^2\left(4\lambda^4 - 11\lambda^2 + 15\right)\mathbf{E}(\mu) - \left(2\lambda^4 - 3\lambda^2 + 9\right)\mathbf{K}(\mu)\right\}, \end{aligned}$$

where $\mu = \sqrt{(\lambda^2 - 1)/\lambda^2}$.

Corollary 4 If X is a random variable having the pdf (1.5) with $\nu = 5$ then

$$\begin{aligned} E(X) &= I_1 + \frac{2\sqrt{5}\lambda\left(9\lambda^6 - 16\delta\lambda^5 - 21\lambda^4 + 64\delta\lambda^3 - 41\lambda^2 + 5\right)}{9\pi\left(\lambda^2-1\right)^4}, \\ E\left(X^2\right) &= \frac{5}{3}, \\ E\left(X^3\right) &= I_3 + \frac{10\sqrt{5}\lambda\left(\lambda^2 + 4\delta\lambda + 5\right)}{3\pi(1+\delta\lambda)^4}, \\ E\left(X^4\right) &= 25, \end{aligned}$$

where $\delta = \text{sign}(\lambda)$.

Corollary 5 If X is a random variable having the pdf (1.5) with $\nu = 6$ then

$$\begin{aligned} E(X) = \frac{3\sqrt{6}\lambda^2}{64\left(\lambda^2-1\right)^5}\Big\{ &\left(8\lambda^8 - 43\lambda^6 + 108\lambda^4 + 65\lambda^2 - 10\right)\mathbf{E}(\mu) \\ &- \left(4\lambda^6 - 21\lambda^4 + 150\lambda^2 - 5\right)\mathbf{K}(\mu)\Big\}, \end{aligned}$$

$$
\begin{aligned}
E\left(X^2\right) &= \frac{3}{2}, \\
E\left(X^3\right) &= \frac{3\sqrt{6}\lambda^2}{32\left(\lambda^2-1\right)^5}\Big\{\left(16\lambda^8-76\lambda^6+131\lambda^4-410\lambda^2-45\right)\mathbf{E}(\mu) \\
&\qquad -\left(8\lambda^6-37\lambda^4-70\lambda^2-285\right)\mathbf{K}(\mu)\Big\}, \\
E\left(X^4\right) &= \frac{27}{2}, \\
E\left(X^5\right) &= \frac{9\sqrt{6}}{16\left(\lambda^2-1\right)^5}\Big\{\lambda^2\left(64\lambda^8-304\lambda^6+569\lambda^4-410\lambda^2+465\right)\mathbf{E}(\mu) \\
&\qquad -\left(32\lambda^8-148\lambda^6+305\lambda^4-30\lambda^2+225\right)\mathbf{K}(\mu)\Big\},
\end{aligned}
$$

where $\mu=\sqrt{(\lambda^2-1)/\lambda^2}$.

Corollary 6 If X is a random variable having the pdf (1.5) with $\nu=7$ then

$$
\begin{aligned}
E\left(X\right) &= I_1+\frac{2\sqrt{7}\lambda}{75\pi\left(\lambda^2-1\right)^6}\Big\{75\lambda^{10}-144\delta\lambda^9-275\lambda^8+672\delta\lambda^7+390\lambda^6 \\
&\qquad -1808\delta\lambda^5+1314\lambda^4-257\lambda^2+33\Big\}, \\
E\left(X^2\right) &= \frac{7}{5}, \\
E\left(X^3\right) &= I_3+\frac{14\sqrt{7}\lambda}{75\pi\left(\lambda^2-1\right)^6}\Big\{15\lambda^{10}-115\lambda^8+48\delta\lambda^7+270\lambda^6+32\delta\lambda^5 \\
&\qquad -990\lambda^4+1200\delta\lambda^3-493\lambda^2+33\Big\}, \\
E\left(X^4\right) &= \frac{49}{5}, \\
E\left(X^5\right) &= I_5+\frac{98\sqrt{7}\lambda\left(3\lambda^4+18\delta\lambda^3+44\lambda^2+54\delta\lambda+33\right)}{15\pi\left(1+\delta\lambda\right)^6}, \\
E\left(X^6\right) &= 343,
\end{aligned}
$$

where $\delta=\text{sign}(\lambda)$.

Corollary 7 If X is a random variable having the pdf (1.5) with $\nu=8$ then

$$
\begin{aligned}
E\left(X\right) &= \frac{5\sqrt{2}\lambda^2}{384\left(\lambda^2-1\right)^7}\Big\{\left(48\lambda^{12}-352\lambda^{10}+1167\lambda^8-2549\lambda^6-1771\lambda^4\right. \\
&\qquad \left.+441\lambda^2-56\right)\mathbf{E}(\mu)
\end{aligned}
$$

$$-\Big(24\lambda^{10}-173\lambda^8+563\lambda^6-3675\lambda^4+217\lambda^2-28\Big)\mathbf{K}(\mu)\Big\},$$

$$E\left(X^2\right) = \frac{4}{3},$$

$$E\left(X^3\right) = \frac{\sqrt{2}\lambda^2}{16\left(\lambda^2-1\right)^7}\Big\{\Big(32\lambda^{12}-216\lambda^{10}+603\lambda^8-731\lambda^6+4767\lambda^4 +735\lambda^2-70\Big)\mathbf{E}(\mu) -\Big(16\lambda^{10}-106\lambda^8+289\lambda^6+1561\lambda^4+3395\lambda^2-35\Big)\mathbf{K}(\mu)\Big\},$$

$$E\left(X^4\right) = 8,$$

$$E\left(X^5\right) = \frac{\sqrt{2}\lambda^2}{6\left(\lambda^2-1\right)^7}\Big\{\Big(128\lambda^{12}-864\lambda^{10}+2482\lambda^8-4009\lambda^6-1617\lambda^4 -10955\lambda^2-525\Big)\mathbf{E}(\mu) -\Big(64\lambda^{10}-424\lambda^8+1191\lambda^6-3941\lambda^4-6475\lambda^2-5775\Big)\mathbf{K}(\mu)\Big\},$$

$$E\left(X^6\right) = 160,$$

$$E\left(X^7\right) = \frac{4\sqrt{2}}{3\left(\lambda^2-1\right)^7}\Big\{\lambda^2\Big(768\lambda^{12}-5184\lambda^{10}+14892\lambda^8-23529\lambda^6+23443\lambda^4 -3675\lambda^2+8645\Big)\mathbf{E}(\mu) -\Big(384\lambda^{12}-2544\lambda^{10}+7146\lambda^8-10521\lambda^6+14735\lambda^4 +2485\lambda^2+3675\Big)\mathbf{K}(\mu)\Big\},$$

where $\mu=\sqrt{(\lambda^2-1)/\lambda^2}$.

Corollary 8 If X is a random variable having the pdf (1.5) with $\nu=9$ then

$$E\left(X\right) = I_1+\frac{6\lambda}{245\pi\left(\lambda^2-1\right)^8}\Big\{245\lambda^{14}-512\delta\lambda^{13}-1225\lambda^{12}+3072\delta\lambda^{11}+2597\lambda^{10} -8704\delta\lambda^9-3073\lambda^8+20480\delta\lambda^7-16057\lambda^6+3949\lambda^4-865\lambda^2+93\Big\},$$

$$E\left(X^2\right) = \frac{9}{7},$$

$$E\left(X^3\right) = I_3+\frac{18\lambda}{1225\pi\left(\lambda^2-1\right)^8}\Big\{525\lambda^{14}-5425\lambda^{12}+3072\delta\lambda^{11}+17885\lambda^{10}-11776\delta\lambda^9 -28105\lambda^8-16384\delta\lambda^7+151823\lambda^6-189952\delta\lambda^5+87221\lambda^4-9721\lambda^2+837\Big\},$$

$$
\begin{aligned}
E\left(X^4\right) &= \frac{243}{35}, \\
E\left(X^5\right) &= I_5 + \frac{162\lambda}{245\pi\left(\lambda^2-1\right)^8}\Big\{63\lambda^{14} - 539\lambda^{12} + 2191\lambda^{10} - 512\delta\lambda^9 - 4739\lambda^8 \\
&\qquad +4096\delta\lambda^7 - 4325\lambda^6 + 17920\delta\lambda^5 - 29561\lambda^4 + 21504\delta\lambda^3 - 6467\lambda^2 + 279\Big\}, \\
E\left(X^6\right) &= \frac{729}{7}, \\
E\left(X^7\right) &= I_7 + \frac{4374\lambda\left(5\lambda^6 + 40\delta\lambda^5 + 139\lambda^4 + 272\delta\lambda^3 + 323\lambda^2 + 232\delta\lambda + 93\right)}{35\pi\left(1+\delta\lambda\right)^8}, \\
E\left(X^8\right) &= 6561,
\end{aligned}
$$

where $\delta = \text{sign}(\lambda)$.

Corollary 9 If X is a random variable having the pdf (1.5) with $\nu = 10$ then

$$
\begin{aligned}
E\left(X\right) &= \frac{35\sqrt{10}\lambda^2}{16384\left(\lambda^2-1\right)^9}\Big\{\big(128\lambda^{16} - 1192\lambda^{14} + 5067\lambda^{12} - 13324\lambda^{10} + 26930\lambda^8 \\
&\qquad +20268\lambda^6 - 6357\lambda^4 + 1392\lambda^2 - 144\big)\mathbf{E}(\mu) \\
&\qquad -\big(64\lambda^{14} - 588\lambda^{12} + 2463\lambda^{10} - 6380\lambda^8 + 39690\lambda^6 - 3096\lambda^4 \\
&\qquad +687\lambda^2 - 72\big)\mathbf{K}(\mu)\Big\}, \\
E\left(X^2\right) &= \frac{5}{4}, \\
E\left(X^3\right) &= \frac{25\sqrt{10}\lambda^2}{8192\left(\lambda^2-1\right)^9}\Big\{\big(256\lambda^{16} - 2240\lambda^{14} + 8550\lambda^{12} - 17819\lambda^{10} + 13720\lambda^8 \\
&\qquad -201834\lambda^6 - 35070\lambda^4 + 5565\lambda^2 - 504\big)\mathbf{E}(\mu) \\
&\qquad -\big(128\lambda^{14} - 1104\lambda^{12} + 4143\lambda^{10} - 8440\lambda^8 - 82710\lambda^6 - 143892\lambda^4 \\
&\qquad +2751\lambda^2 - 252\big)\mathbf{K}(\mu)\Big\}, \\
E\left(X^4\right) &= \frac{25}{4}, \\
E\left(X^5\right) &= \frac{25\sqrt{10}\lambda^2}{4096\left(\lambda^2-1\right)^9}\Big\{\big(1024\lambda^{16} - 8960\lambda^{14} + 34704\lambda^{12} - 78521\lambda^{10} + 124054\lambda^8 \\
&\qquad +323010\lambda^6 + 706944\lambda^4 + 47775\lambda^2 - 3150\big)\mathbf{E}(\mu) \\
&\qquad -\big(512\lambda^{14} - 4416\lambda^{12} + 16824\lambda^{10} - 37351\lambda^8 + 235404\lambda^6 + 559062\lambda^4 \\
&\qquad +378420\lambda^2 - 1575\big)\mathbf{K}(\mu)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
E\left(X^6\right) &= \frac{625}{8}, \\
E\left(X^7\right) &= \frac{125\sqrt{10}\lambda^2}{2048\left(\lambda^2-1\right)^9}\Big\{\Big(2048\lambda^{16}-17920\lambda^{14}+69408\lambda^{12}-155992\lambda^{10}+222383\lambda^8 \\
&\qquad -367020\lambda^6-470022\lambda^4-418740\lambda^2-11025\Big)\mathbf{E}(\mu) \\
&\qquad -\Big(1024\lambda^{14}-8832\lambda^{12}+33648\lambda^{10}-74177\lambda^8+48708\lambda^6-481446\lambda^4 \\
&\qquad -478380\lambda^2-187425\Big)\mathbf{K}(\mu)\Big\}, \\
E\left(X^8\right) &= \frac{21875}{8}, \\
E\left(X^9\right) &= \frac{625\sqrt{10}}{1024\left(\lambda^2-1\right)^9}\Big\{\lambda^2\Big(16384\lambda^{16}-143360\lambda^{14}+555264\lambda^{12}-1247936\lambda^{10} \\
&\qquad +1790089\lambda^8-1659780\lambda^6+1416534\lambda^4+164220\lambda^2+255465\Big)\mathbf{E}(\mu) \\
&\qquad -\Big(8192\lambda^{16}-70656\lambda^{14}+269184\lambda^{12}-593416\lambda^{10}+841689\lambda^8-589428\lambda^6 \\
&\qquad +971670\lambda^4+210420\lambda^2+99225\Big)\mathbf{K}(\mu)\Big\},
\end{aligned}
$$

where $\mu = \sqrt{(\lambda^2-1)/\lambda^2}$.

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On the DCP Property of Exponential Partial Sums

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Abstract

In [6] we proved that the functions

$$(*) \qquad \sum_{k=1}^{\infty} a_k \frac{(1+z)^k}{k!}, \quad 0 \neq a_1 \geq a_2 \geq \cdots \geq 0,$$

are convex univalent in the unit disk $\mathbb{D}$, which extended previous results of Suffridge [8]. In the present paper we prove a conjecture made in [6], namely, that the functions $(*)$, under the further restriction that $0 \neq a_1 = a_2$, even belong to the much narrower class DCP of the functions which, under Hadamard product, preserve direction-convexity of univalent functions in the unit disk $\mathbb{D}$.

Keywords

DCP, Direction-Convexity-Preserving-Functions, Univalent Functions, Hadamard Product, Computer Algebra.

1. Introduction

This paper as well as [6] were inspired by Ted J. Suffridge [8] who studied the partial sums

$$Q_n(z) = \sum_{k=0}^{n} \frac{(1+z)^k}{k!}, \quad z \in \mathbb{C}, \quad n \in \mathbb{N}, \tag{1.1}$$

of the series $e^{1+z} = \sum_{k=0}^{\infty}(1+z)^k/k!$. The main result in [8] was that the Q_n are in the class $\mathcal{K}$ of convex univalent functions in the unit disk $\mathbb{D}$. Note that $Q_n' = Q_{n-1}$, so that all derivatives of Q_n are as well convex univalent or constants which made this system of functions particularly interesting.

In [6] we studied convex combinations of the Q_n, and obtained the following extension of Suffridge's result.

Theorem 1. *For $0 \neq a_1 \geq a_2 \geq \cdots \geq 0$ we have*

$$f(z) := \sum_{k=1}^{\infty} a_k \frac{(1+z)^k}{k!} \in \mathcal{K}. \tag{1.2}$$

Note that this convex set of functions is (essentially) also invariant with respect to differentiation.

In the present paper we shall establish a conjecture about the functions (1.1) made in [6] which asserts that (most of) those functions actually belong to a much smaller class of functions, called DCP, contained in the set of convex univalent functions in $\mathbb{D}$. We recall the definition of the DCP functions.

Let $\mathcal{A}$ denote the set of analytic functions in $\mathbb{D}$, $f * g$ the Hadamard product or convolution between to members of $\mathcal{A}$, and $\mathcal{K} \subset \mathcal{A}$ the set of convex univalent functions in $\mathbb{D}$. A domain $\Omega \subset \mathbb{C}$ is said to be *convex in the direction* $e^{i\varphi}$, $\varphi \in \mathbb{R}$, if and only if for every $a \in \mathbb{C}$ the set

$$\Omega \cap \left\{a + te^{i\varphi} \,:\, t \in \mathbb{R}\right\}$$

is either connected or empty. Accordingly we define the classes $\mathcal{K}(\varphi) \subset \mathcal{A}$, $\varphi \in \mathbb{R}$, of the functions *convex in the direction* $e^{i\varphi}$ as

$$\mathcal{K}(\varphi) := \left\{ f \in \mathcal{A} : \ f \text{ univalent and } f(\mathbb{D}) \text{ convex in the direction } e^{i\varphi} \right\}.$$

Finally, a function $g \in \mathcal{A}$ is called *Direction-Convexity-Preserving* ($g \in DCP$) if and only if

$$g * f \in K(\varphi) \text{ for all } f \in K(\varphi) \text{ and all } \varphi \in \mathbb{R}.$$

Functions in DCP have many other intriguing convolution-type properties, for instance the preservation of convex harmonic functions in $\mathbb{D}$, and of Jordan curves in the plane with convex interior domain; we refer to [4], [5] for more details. Note that, by definition, we have $\cap_{\varphi\in\mathbb{R}}\mathcal{K}(\varphi) = \mathcal{K}$ so that $g \in DCP$ preserves, under convolution, the class $\mathcal{K}$ as well, which by the results in [7], obtained in the context of the former Pólya-Schoenberg conjecture, implies that $DCP \subset \mathcal{K}$. Actually, DCP is much smaller than $\mathcal{K}$. The following result has been conjectured in [6] and sharpens Theorem 1.

Theorem 2. *For* $0 \neq a_1 = a_2 \geq a_3 \geq \cdots \geq 0$ *we have*

$$g(z) := \sum_{k=1}^{\infty} a_k \frac{(1+z)^k}{k!} \in DCP. \tag{1.3}$$

The special case $a_k = 1,\ k \in \mathbb{N}$, namely $g(z) = e^{1+z} - 1 \in DCP$ has first been established by G. Kurth [2]. Another proof can be found in [6].

The condition $a_1 = a_2$ cannot be just dropped, as it is readily verified that, for instance,

$$2\frac{1+z}{1!} + \frac{(1+z)^2}{2!} + \frac{(1+z)^3}{3!} \notin DCP. \tag{1.4}$$

However there may be weaker conditions than $a_1 = a_2$ which do the same job.

The class DCP is somewhat special in the following sense: for $f \in DCP$ we do not necessarily have $f_r(z) := f(rz) \in DCP$ for $0 < r < 1$. There are very

good reasons, however, to assume that the following generalization of Theorem 2 is valid.

Conjecture 1. *For* $0 \neq a_1 = a_2 \geq a_3 \geq \cdots \geq 0$ *and* $0 < r \leq 1$ *we have*

$$\sum_{k=1}^{\infty} a_k \frac{(1+rz)^k}{k!} \in DCP. \tag{1.5}$$

Let ${}_1F_1$ be the confluent hypergeometric function, so that

$${}_1F_1(a, b, 1+z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{(1+z)^k}{k!}.$$

An immediate consequence of Theorem 1 is

Corollary 1. *For* $0 < a \leq b$ *we have* ${}_1F_1(a, b, 1+z) \in \mathcal{K}$.

However, because of the restriction $a_1 = a_2$ in Theorem 2, we cannot conclude the corresponding result with $\mathcal{K}$ replaced by DCP. Nevertheless, there is some evidence for this to hold anyway.

Conjecture 2. *For* $0 < a \leq b$ *we have* ${}_1F_1(a, b, 1+z) \in DCP$.

The rest of this paper is devoted to the proof of Theorem 2. It is highly computational, using computer algebra and numerical evaluations. However, the computer algebra part is restricted to the manipulation of trigonometric expressions, rearrangements of polynomials of modest degree (≤ 22) with rational coefficients, and numerical work to the identification of maxima and minima of those polynomials in the unit interval $[0, 1]$. These calculations have been performed using the software package Mathematica 4.1 [3], applying high-precision mode for the numerics. The scripts of these calculations, in form of Mathematica notebooks, are available from the authors upon request. In the sequel we shall not comment on this approach any more.

2. Proof of Theorem 2

The following criterion for membership in DCP is a slight variant of [4, Theorem 4]:

Lemma 1. *Let $g \in \mathcal{K}$ be analytic in $\overline{\mathbb{D}}$ and let $v(x) := \operatorname{Re} g(e^{ix})$, $x \in \mathbb{R}$. Then $g \in DCP$ if and only if*

$$\sigma_v(x) := (v''(x))^2 - v'(x)\, v'''(x) \geq 0\,, \quad x \in \mathbb{R}\,. \tag{2.1}$$

Proof of Theorem 2. Let g be the function in (1.3). Then g is entire and, by Theorem 1, in $\mathcal{K}$. Therefore we only have to verify that the function $v(x) := \operatorname{Re} g(e^{ix})$ satisfies (2.1). Writing

$$u_j(x) := \operatorname{Re} \sum_{k=1}^{j} \frac{(1+e^{ix})^k}{k!}, \quad j = 2, 3, \ldots, \quad x \in \mathbb{R}\,, \tag{2.2}$$

we find that

$$v(x) = \sum_{j=2}^{\infty} \lambda_k u_j(x) \quad \text{with} \quad \lambda_j := a_j - a_{j+1} \geq 0 \quad \text{and} \quad a_2 = \sum_{j=2}^{\infty} \lambda_j > 0\,. \tag{2.3}$$

Condition (2.1) is then equivalent to

$$\sum_{j,k=2}^{\infty} \lambda_j \lambda_k u_{j,k}(x) \geq 0, \quad x \in \mathbb{R}\,, \tag{2.4}$$

with

$$u_{jk}(x) := u_j''(x) u_k''(x) - \frac{1}{2}(u_j'(x) u_k'''(x) + u_k'(x) u_j'''(x)), \tag{2.5}$$

for $j, k = 2, 3, 4, \ldots$ and $x \in \mathbb{R}$. This should hold for arbitrary coefficients $\lambda_k \geq 0$ with $\sum_{k=2}^{\infty} \lambda_k > 0$, and thus, by definition, if and only if the doubly-infinite matrices $\{u_{j,k}(x)\}_{j,k\geq 2}$ are co-positive for all $x \in \mathbb{R}$. In the subsequent sections we shall show that

$$u_{jk}(x) \geq 0, \quad j, k \geq 2, \quad x \in \mathbb{R}\,, \tag{2.6}$$

which will complete the proof of Theorem 2. ∎

It should be observed that actually $u_{1,j}(x) \geq 0$ does not hold generally for $j \geq 2$. This does not contradict *a prori* Conjecture 2 since in those cases only specific systems $\{\lambda_k\}$ are involved.

3. Preliminaries I

3.1. For the functions $u_j,\ j \in \mathbb{N}$, of (2.2) we find

$$u_j(x) = \sum_{k=1}^{j} \frac{2^k}{k!} T_k(\nu)\nu^k = p_{j,0}(\nu^2), \tag{3.1}$$

where T_k denotes the Chebychev polynomial of degree k, while $\nu = \nu(x) := \cos(\frac{x}{2})$, and $p_{j,0}$ is some polynomial. Using the relations

$$(\nu')^2 = \frac{1-\nu^2}{4}, \quad \nu'' = -\frac{\nu}{4}, \tag{3.2}$$

we find polynomials $p_{j,s},\ j \in \mathbb{N},\ s = 1, 2, 3$, such that

$$u_j'(x) = \nu\nu' p_{j,1}(\nu^2), \quad u_j''(x) = p_{j,2}(\nu^2), \quad u_j'''(x) = \nu\nu' p_{j,3}(\nu^2), \tag{3.3}$$

This shows, in particular, that all terms in the expressions $u_{j,k}$, and therefore $u_{j,k}$ itself, are polynomials in the variable ν^2, and the proof of (2.6) has to be given for $\nu \in [0, 1]$ only.

3.2. When dealing with large values of j it will be convenient to write $u_j(x) = u(x) - r_j(x)$, with

$$u(x) := \operatorname{Re} e^{1+e^{ix}}, \qquad r_j(x) := \operatorname{Re} \sum_{k=j+1}^{\infty} \frac{(1+e^{ix})^k}{k!}.$$

Using

$$\mu = \mu(x) := |e^{1+e^{ix}}|$$

we obtain

$$\begin{aligned}
\frac{u'(x)}{\mu} &= (2\nu^2-1)\sin(4\nu\nu') + 4\nu\nu'\cos(4\nu\nu')\,,\\
\frac{u''(x)}{\mu} &= 4(4\nu^2-1)\nu\nu'\sin(4\nu\nu') - \nu^2(8\nu^2-6)\cos(4\nu\nu')\,,\\
\frac{u'''(x)}{\mu} &= -(4\nu^2+1)\left[(1-8\nu^2+8\nu^4)\sin(4\nu\nu')\right.\\
&\qquad \left. +8\nu\nu'(2\nu^2-1)\cos(4\nu\nu')\right].
\end{aligned} \tag{3.4}$$

Together with (3.3) we thus get for

$$u_{j,\infty} := u''u_j'' - \frac{1}{2}\left(u'u_j''' + u'''u_j'\right)$$

the representations

$$\frac{u_{j,\infty}}{\mu} = p_{j,1}^*(\nu^2)\nu\nu'\sin(4\nu\nu') + p_{j,2}^*(\nu^2)\cos(4\nu\nu'), \tag{3.5}$$

with $p_{j,1}^*, p_{j,2}^*$ polynomials. Furthermore, for $u_{\infty,\infty} := (u'')^2 - u'u'''$, we find

$$\begin{aligned}
\frac{u_{\infty,\infty}}{\mu^2} &= -\tfrac{1}{2} + \nu^2 + 4\nu^4 + \left(\tfrac{1}{2} - \nu^2(3-4\nu^2)^2\right)\cos(8\nu\nu')\\
&\qquad +2\nu\nu'(3-16\nu^2+16\nu^4)\sin(8\nu\nu')
\end{aligned} \tag{3.6}$$

3.3. In the sequel we shall approximate the trigonometric functions appearing explicitly in (3.5),(3.6) by their Taylor expansions with remainder:

$$\begin{aligned}
\cos(y) &= \sum_{k=0}^{n-1}(-1)^k\frac{y^{2k}}{(2k)!} + a\,(-1)^n\frac{y^{2n}}{(2n)!}\,,\\
\sin(y) &= \sum_{k=0}^{n-1}(-1)^k\frac{y^{2k+1}}{(2k+1)!} + b\,(-1)^n\frac{y^{2n+1}}{(2n+1)!}\,,
\end{aligned} \tag{3.7}$$

with $y \in \mathbb{R}$, $n \in \mathbb{N}$, where $a = a(y)$ and $b = b(y)$ are the error terms. We shall apply these formulas only for $y = 4\nu\nu'$ and $y = 4\nu\nu'$, i.e. for values of y with $|y| \le 2$. In these cases the expansions in (3.7) are (essentially) of the Leibniz type and this implies that we have

$$0 \le a_n(y), b_n(y) \le 1. \tag{3.8}$$

We shall make use of these approximations in the sense that we carry a, b along as if they were independent of y, but only restricted by (3.8)

4. Preliminaries II

We shall need estimates for the expressions $|u^{(k)}|$, $|u_j^{(k)}|, |r_j^{(k)}|$. They will be given in this section.

4.1. In Sect. 5.1 we make use of the following estimates:

$$|u'(x)| \le \frac{8}{5}\mu|\nu|^3, \quad |u''(x)| \le 3\mu|\nu|^2, \quad |u'''(x)| \le 5\mu|\nu|. \tag{4.1}$$

We prove the first one of these, using approximations from (3.7) (with $n = 4$ for cos and $n = 3$ for sin). Inserting this into (3.4) we get

$$\begin{aligned}\left|\frac{u'(x)}{\mu\nu^3}\right| &= \tfrac{8}{315}\,|\nu'(315 - 14\nu^4(1-\nu^2)^3 - 210(1-\nu^4) + 42(2\nu^2+\nu^4)(1-\nu^2)^2) \\ &\qquad +a_4\nu'\nu^6(1-\nu^2)^4 - 2b_3\nu'\nu^4(1-\nu^2)^3(2\nu^2-1)| \\ &= \tfrac{8}{315}|\nu'Q + a_4q_1 + b_3q_2|, \quad \text{say.}\end{aligned}$$

We easily estimate

$$|q_1| = |\nu'\nu^6(1-\nu^2)^4| \le \tfrac{1}{2}(\nu^2(1-\nu^2))^3 \le \tfrac{1}{128}\,,$$

$$|q_2| = |2\nu'\nu^4(1-\nu^2)^3(2\nu^2-1)| \le (\nu^2(1-\nu^2))^2 \le \tfrac{1}{16}\,,$$

so that we are left with a proof for

$$|\nu'Q| \le 63 - \frac{1}{128} - \frac{1}{16}.$$

Taking squares and using (3.2) we find that

$$(1-\nu^2)\left(105 + 84\nu^2 + 70\nu^4 + 42\nu^6 + 14\nu^{10}\right)^2 \le 15840, \quad 0 \le \nu \le 1, \tag{4.2}$$

is a sufficient condition for the truth of the first inequality in (4.1). This is not critical at all and has been verified numerically. The remaining two estimates in (4.1) can be derived in the same fashion.

4.2. In Sect. 5.2 we shall need upper bounds $b_{j,k}$ in

$$|u_j^{(k)}(x)| \le b_{j,k}, \quad j \le 5,\ k = 1, 2, 3.$$

Using the method mentioned in Sect. 3.1 we obtain the needed representations for the $u_j^{(k)}$ and the bounds $b_{j,k}$. This information is collected in Table 1. The given bounds $b_{j,k}$ have been numerically verified.

4.3. To obtain suitable bounds for the $r_j(x)$ we proceed as follows: define

$$R_j(x) := \sum_{k=j+1}^{\infty} \frac{(1+e^{ix})^k}{k!}, \quad j = 2, 3, \ldots, \tag{4.3}$$

and note that

$$R_j(x) = \frac{w^{j+1}}{(j+1)!}\ {}_1F_1(1, j+2, w)\,, \quad w = 1 + e^{ix},$$

where ${}_1F_1$ denotes the confluent hypergeometric function.

Lemma 2. *Let* $v_j(x) := |{}_1F_1(1, j+1, 1+e^{ix})|$. *Then*

$$v_j(x) \le \mu(x)(1 - \frac{43j\nu^2(x)}{50(j+1)}) =: \mu\, d_j(\nu)\,, \quad x \in \mathbb{R}. \tag{4.4}$$

Proof From [1, 13.2.1] we get

$${}_1F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt}\, t^{a-1}\, (1-t)^{b-a-1}\, dt\,, \tag{4.5}$$

for $\operatorname{Re} b > \operatorname{Re} a > 0$. The convexity of the exponential function gives

$$\left| e^{-(1+e^{ix})t} \right| = e^{-(1+\cos x)t} \le 1 - \widetilde{c}_0(1+\cos x)\, t\,,$$

for $x \in \mathbb{R}$, $0 \le t \le 1$, and $\widetilde{c}_0 := \dfrac{1-e^{-2}}{2} = 0.432332\ldots$ Inserting this into (4.5) we obtain:

$$\begin{aligned} v_j(x) \le & \; e^{1+\cos x} \int_0^1 e^{-(1+\cos x)t}\, j\, t^{j-1}\, dt \\ \le & \; \mu \int_0^1 (1 - c_0(1+\cos x)\, t)\; j\, t^{j-1}\, dt \\ = & \; \mu \left(1 - \frac{2c_0 j \nu^2(x)}{j+1}\right), \end{aligned}$$

where $c_0 := \frac{43}{100} < \widetilde{c}_0$. ∎

Since

$$R_j'(x) = R_{j-1}(x)\, i\, e^{ix}\,,$$

$$R_j''(x) = -R_{j-2}(x)\, e^{2ix} - R_{j-1}(x)\, e^{ix}\,,$$

$$R_j'''(x) = R_{j-3}(x)\, e^{4ix} + R_{j-2}(x)\, e^{3ix} - 2\, i\, R_{j-2}(x)\, e^{2ix} - i\, R_{j-1}(x)\, e^{ix}\,,$$

we find, using Lemma 2,

$$\left| r_j^{(k)}(x) \right| \le \left| R_j^{(k)}(x) \right| \le \mu r_{j,k}, \quad k = 1,2,3,\; j \ge 2, \tag{4.6}$$

where

$$\begin{aligned} r_{j,1} \; &:= \frac{(2|\nu|)^j}{j!}\, d_j(\nu)\,, \\ r_{j,2} \; &:= \frac{(2|\nu|)^{j-1}}{(j-1)!} \left[d_{j-1}(\nu) + \frac{2|\nu|}{j}\, d_j(\nu) \right], \qquad (4.7) \\ r_{j,3} \; &:= \frac{(2|\nu|)^{j-2}}{(j-2)!} \left[d_{j-2}(\nu) + \frac{6|\nu|}{j-1}\, d_{j-1}(\nu) + \frac{4|\nu|^2}{j(j-1)}\, d_j(\nu) \right]. \end{aligned}$$

Note that

$$r_{n,k} \le r_{m,k}, \quad n \ge m \ge 4,\; k = 1,2,3\,. \tag{4.8}$$

5. Proof of (2.6)

5.1. **The cases** $j,k \geq 6$**.** Since $u_j(x) = u(x) - r_j(x)$ we have

$$\begin{aligned} u_{j,k}(x) = \; & (u-r_j)''(u-r_k)'' - \\ & \tfrac{1}{2}\left\{(u-r_j)'\,(u-r_k)''' + (u-r_k)'\,(u-r_j)'''\right\} \\ = \; & (u'')^2 - u'u''' - u''(r_j''+r_k'') + r_j''r_k'' + \\ & \tfrac{1}{2}\left\{u'(r_j'''+r_k''') + u'''(r_j'+r_k') - r_j'r_k''' - r_k'r_j'''\right\} \\ \geq \; & (u'')^2 - u'u''' - |u''|(|r_j''|+|r_k''|) - |r_j''||r_k''| - \\ & \tfrac{1}{2}\left[|u'|(|r_j'''|+|r_k'''|) + |u'''|(|r_j'|+|r_k'|) + |r_j'||r_k'''| + |r_k'||r_j'''|\right]\,. \end{aligned}$$

Furthermore, in view of (4.1), (4.7) we may write this inequality in the form:

$$\begin{aligned} \frac{u_{j,k}}{\mu^2} \geq \; & \frac{u_{\infty,\infty}}{\mu^2} - \Big[3\nu^2(r_{j,2}+r_{k,2}) + r_{j,2}r_{k,2} + \frac{4}{5}\nu^3\,(r_{j,3}+r_{k,3}) \\ & + \frac{5}{2}\,\nu\,(r_{j,1}+r_{k,1}) + \frac{1}{2}(r_{j,1}r_{k,3}+r_{k,1}r_{j,3})\Big] \qquad (5.1) \\ =: \; & s_{j,k}. \end{aligned}$$

The $s_{j,k}$ depend on ν only, and in Sect. 3.1 we pointed out that we need to verify $u_{j,k} \geq 0$ for $0 \leq \nu \leq 1$ only. Therefore we can replace $|\nu|$ by ν wherever it appears in (5.1). Taking this into account, and using (4.8), we find that the $s_{j,k}$ are increasing with both, j,k. Hence $u_{j,k} \geq 0$ for $j,k \geq 6$ will follow if $s_{6,6} \geq 0$ holds. Using (3.6), (4.7) and (3.7) with $n=3$ we find the equivalent condition

$$\nu^4(Q(\nu) + a\,q_1(\nu) + b\,q_2(\nu)) \geq 0, \quad \nu,a,b \in [0,1], \qquad (5.2)$$

where

$$\begin{cases} Q(\nu) &= \frac{4}{3}+\frac{8}{5}\,\nu^2-\frac{28}{9}\,\nu^3-]\frac{3736}{75}\,\nu^4+\frac{81346}{39375}\,\nu^5+\frac{1084136}{4725}\,\nu^6-\frac{1616}{118125}\,\nu^7 \\ & \quad -\frac{148115552}{354375}\,\nu^8+\frac{12212}{70875}\,\nu^9+\frac{3326016569}{8859375}\,\nu^{10}-\frac{7396}{118125}\,\nu^{11}-\frac{940859168}{6890625}\,\nu^{12}\,, \\ q_1(\nu) &= -\frac{128}{45}\nu^2(\nu^2-1)^3(2\nu^2-1)(1-16\nu^2+16\nu^4)\,, \\ q_2(\nu) &= -\frac{1024}{315}\nu^4(\nu^2-1)^4(4\nu^2-1)(4\nu^2-3)\,. \end{cases}$$

It is sufficient to show that

$$\min_{0\le\nu\le1} Q(\nu)+\min\left(0,\min_{0\le\nu\le1} q_1(\nu)\right)+\min\left(0,\min_{0\le\nu\le1} q_2(\nu)\right)\ge 0. \tag{5.3}$$

Numerically we find

$$\begin{aligned} \min_{0\le\nu\le1} Q(\nu) &= 0.543527...\,, \\ \min_{0\le\nu\le1} q_1(\nu) &= -0.0644599...\,, \\ \min_{0\le\nu\le1} q_2(\nu) &= -0.0375348...\,, \end{aligned}$$

with all digits significant. (5.3) follows. ∎

5.2. **The cases** $2\le j\le 5$**,** $k\ge 6$**.** We consider again the functions $u_{j,k}$ but now in a slightly different arrangement:

$$\begin{aligned} u_{j,k} = &\ u_j''(u-r_k)''-\tfrac{1}{2}\left(u_j'''(u-r_k)'+u_j'(u-r_k)'''\right) \\ = &\ u_{j,\infty}-u_j''r_k''+\tfrac{1}{2}\left(u_j'''r_k'+u_j'r_k'''\right) \\ \ge &\ u_{j,\infty}-|u_j''||r_k''|-\tfrac{1}{2}\left(|u_j'''||r_k'|+|u_j'|r_k'''|\right). \end{aligned}$$

Hence, using (4.6),(4.7) and Sect. 3.1, we find

$$\frac{u_{j,k}}{\mu}\ge\frac{u_{j,\infty}}{\mu}-b_{j,2}r_{k,2}-\frac{1}{2}\left(b_{j,3}r_{k,1}+b_{j,1}r_{k,3}\right),$$

and since the $r_{k,j}$ are decreasing with k, we need to verify only $A_j\ge 0$ for $j=2,3,4,5$ with

$$A_j:=\frac{u_{j,\infty}}{\mu}-b_{j,2}r_{6,2}-\frac{1}{2}\left(b_{j,3}r_{6,1}+b_{j,1}r_{6,3}\right). \tag{5.4}$$

We proceed as before: to the functions $\frac{u_{j,\infty}}{\mu}$ we apply (3.8) with $n = 2$, and then we are left with expressions of the form

$$A_j = \nu^{c(j)} \left(Q_j(\nu) + a\, q_{1,j}(\nu) + b\, q_{1,j}(\nu) \right),$$

with polynomials $Q_j, q_{1,j}, q_{2,j}$, and numbers a, b depending on x, j but satisfying $0 \le a,\, b \le 1$. Here $c(3) = 6$ and $c(j) = 4$ otherwise. We set

$$M_j := \min_{0\le\nu\le 1} Q_j(\nu), \quad m_{k,j} := \min\left(0, \min_{0\le\nu\le 1} q_{k,j}(\nu)\right), \quad k = 1, 2,$$

and show that $M_j + m_{1,j} + m_{2,j} > 0,\ j = 2, 3, 4, 5$, which implies our assertion concerning A_j. The following expressions for the polynomials as well as the numerical values of the minima were computed using [3].

$$\begin{cases} Q_2 = & 8 - \dfrac{32\nu^3}{5} - \dfrac{1504\nu^4}{15} + \dfrac{32804\nu^5}{7875} + \\ & \dfrac{236028\nu^6}{875} + \dfrac{688\nu^7}{2625} - 384\nu^8 + 384\nu^{10} - \dfrac{512\nu^{12}}{3}\,, \\ q_{1,2} = & -\dfrac{8\nu^2}{3}\left(-1+\nu^2\right)^2\left(3 - 25\nu^2 + 60\nu^4 - 72\nu^6 + 32\nu^8\right)\,, \\ q_{2,2} = & -\dfrac{8\nu^2}{15}\left(-1+\nu^2\right)^3\left(3 - 24\nu^2 + 72\nu^4 - 112\nu^6 + 64\nu^8\right)\,, \\ M_2 = & 0.367404\,, \quad m_{1,2} = -0.245545\,, \quad m_{2,2} = -0.0333333\,. \end{cases}$$

Thus $A_2 \ge M_2 + m_{1,2} + m_{2,2} > 0$.

$$\begin{cases} Q_3 = & -\dfrac{8}{7875}\left(-10500-99750\nu^2+19250\nu^3+\right. \\ & 429800\nu^4-13017\nu^5-899595\nu^6-516\nu^7+ \\ & \left.1155000\nu^8-924000\nu^{10}+336000\nu^{12}\right)\,, \\ q_{1,3} = & -\dfrac{32\nu^2}{3}\left(-1+\nu^2\right)^2\left(5-27\nu^2+49\nu^4-44\nu^6+16\nu^8\right)\,, \\ q_{2,3} = & -\dfrac{32\nu^2}{15}\left(-1+\nu^2\right)^3\left(5-28\nu^2+66\nu^4-72\nu^6+32\nu^8\right)\,, \\ M_3 = & 7.44432\,, \quad m_{1,3}=-2.38551\,, \quad m_{2,3}=0\,. \end{cases}$$

Thus $A_3 \geq M_3 + m_{1,3} + m_{2,3} > 0$.

$$\begin{cases} Q_4 = & \dfrac{4}{3}-\dfrac{1549\nu^3}{225}-\dfrac{28202\nu^4}{675}+\dfrac{75766\nu^5}{16875}+ \\ & \dfrac{236101\nu^6}{630}+\dfrac{3698\nu^7}{13125}-\dfrac{9280\nu^8}{9}+\dfrac{16960\nu^{10}}{9}- \\ & \dfrac{19712\nu^{12}}{9}+\dfrac{13312\nu^{14}}{9}-\dfrac{4096\nu^{16}}{9}\,, \\ q_{1,4} = & -\dfrac{4\nu^2}{9}\left(-1+\nu^2\right)^2\left(3-25\nu^2+\right. \\ & \left.396\nu^4-1528\nu^6+2272\nu^8-1664\nu^{10}+512\nu^{12}\right)\,, \\ q_{2,4} = & -\dfrac{4\nu^2}{45}\left(-1+\nu^2\right)^3\left(3-24\nu^2+\right. \\ & \left.408\nu^4-1712\nu^6+3264\nu^8-2816\nu^{10}+1024\nu^{12}\right)\,, \\ M_4 = & 0.893447\,, \quad m_{1,4}=-0.315326\,, \quad m_{2,4}=0\,. \end{cases}$$

Thus $A_4 \geq M_4 + m_{1,4} + m_{2,4} > 0$.

$$\begin{cases} Q_5 = & \frac{4}{3}+\frac{8\nu^2}{3}-\frac{178\nu^3}{25}+\frac{18104\nu^4}{3375}+\frac{400976\nu^5}{84375}- \\ & \frac{1899013\nu^6}{13125}+\frac{4988\nu^7}{21875}+\frac{2216\nu^8}{3}-1680\nu^{10}+ \\ & 2752\nu^{12}-\frac{8704\nu^{14}}{3}+\frac{5120\nu^{16}}{3}-\frac{4096\nu^{18}}{9}\,, \\ q_{1,5} = & -\frac{4\nu^2}{9}\left(-1+\nu^2\right)^2\left(3+5\nu^2-102\nu^4+\right. \\ & \left. 798\nu^6-2408\nu^8+3072\nu^{10}-1920\nu^{12}+512\nu^{14}\right)\,, \\ q_{2,5} = & -\frac{4\nu^2}{45}\left(-1+\nu^2\right)^3\left(3+6\nu^2-96\nu^4+\right. \\ & \left. 860\nu^6-2864\nu^8+4608\nu^{10}-3328\nu^{12}+1024\nu^{14}\right)\,, \\ M_5 = & 1.09764\,, \quad m_{1,5}=-0.23118\,, \quad m_{2,5}=0\,. \end{cases}$$

Thus $A_5 \geq M_5 + m_{1,5} + m_{2,5} > 0$.

5.3. **The cases** $2 \leq j \leq k \leq 5$**.** As mentioned before, the $u_{j,k}$ are all even polynomials in the variable $\nu = \nu(x)$, which are easily determined by the method explained in Sect. 3.1. They are listed in Table 2, along with the numerical (positive!) minima $m_{j,k}$ of the functions $u^*_{j,k} := u_{j,k}(x)/\nu(x)^c$, $0 \leq x \leq \pi$ where $c = c(j,k)$ has been choosen to eliminate the zero at $x = \pi$. This completes the proof of (2.6).

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Table 1. Upper bounds for the derivatives $(u_j^{(k)})^2$.

j	k	$(u_j^{(k)})^2$	$b_{j,k}$
2	1	$64\nu^6(1-\nu^2)$	$8\nu^3$
2	2	$16\nu^4(3-4\nu^2)^2$	$12\nu^2$
2	3	$16\nu^2(1-\nu^2)(3-8\nu^2)^2$	12ν
3	1	$256\nu^{10}(1-\nu^2)$	$16\nu^5$
3	2	$64\nu^8(5-6\nu^2)^2$	$40\nu^4$
3	3	$256\nu^6(1-\nu^2)(5-9\nu^2)^2$	$80\nu^3$
4	1	$\frac{16}{9}\nu^6(1-\nu^2)(1+16\nu^4)^2$	$\frac{34}{5}\nu^3$
4	2	$\frac{4}{9}\nu^4(3-4\nu^2+112\nu^4-128\nu^6)^2$	$\frac{34}{3}\nu^2$
4	3	$\frac{4}{9}\nu^2(1-\nu^2)(3-8\nu^2+336\nu^4-512\nu^6)^2$	$\frac{112}{5}\nu$
5	1	$\frac{16}{9}\nu^6(1-\nu^2)(1+3\nu^2+16\nu^6)^2$	$\frac{174}{25}\nu^3$
5	2	$\frac{4}{9}\nu^4(3+11\nu^2-18\nu^4+144\nu^6-160\nu^8)^2$	$\frac{40}{3}\nu^2$
5	3	$\frac{4}{9}\nu^2(1-\nu^2)(3+22\nu^2-54\nu^4+576\nu^6-800\nu^8)^2$	28ν

Table 2. The cases $2 \leq j \leq k \leq 5$.

j	k	$u_{j,k}(x)$	$m_{j,k}$
2	2	$16\nu^4(3-2\nu^2)$	16
2	3	$32\nu^6\,(2+\nu^2-2\nu^4)$	32
2	4	$\frac{8}{3}\nu^4\,(3-2\nu^2-48\nu^4+192\nu^6-128\nu^8)$	5.27605
2	5	$\frac{8}{3}\nu^4\,(3+4\nu^2+3\nu^4-198\nu^6+496\nu^8-288\nu^{10})$	5.72455
3	3	$64\nu^8\,(5-4\nu^2)$	8
3	4	$\frac{16}{3}\nu^6\,(2+\nu^2+62\nu^4-16\nu^6-32\nu^8)$	$\frac{32}{3}$
3	5	$\frac{4}{9}\nu^4\,(3-2\nu^2-96\nu^4+384\nu^6+1536\nu^8-1536\nu^{10})$	$\frac{32}{3}$
4	4	$\frac{4}{9}\nu^4\,(3-2\nu^2-96\nu^4+384\nu^6+1536\nu^8-1536\nu^{10})$	1.0315
4	5	$\frac{4}{9}\nu^4\,(3+4\nu^2-45\nu^4+186\nu^6+320\nu^8+1152\nu^{10}$ $-768\nu^{12}-512\nu^{14})$	$\frac{4}{3}$
5	5	$\frac{4}{9}\nu^4\,(3+10\nu^2+51\nu^4-432\nu^6+896\nu^8+384\nu^{10}$ $+1536\nu^{12}-2048\nu^{14})$	$\frac{4}{3}$

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Improvements of some Ostrowski type inequalities

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Abstract. A connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity is established. Using this connection, certain strict improvements of some Ostrowski type inequalities are obtained.

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1. Introduction

The following Ostrowski inequality is well known [8]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\left\|f'\right\|_\infty .$$

It holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b) i.e.

$$\left\|f'\right\|_\infty := \sup_{t\in(a,b)} \left|f'(t)\right| < +\infty.$$

Ostrowski proved this inequality in 1938.and since then it has been generalized in a number of ways. Also over the last few years some new inequalities of this type have been intensively considered together with their applications in Numerical analysis and Probability.

Recently Dragomir and Barnet [5] proved the following result:

Theorem A. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b) whose second derivative $f'':(a,b)\to\mathbb{R}$ is bounded on (a,b). Then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\&\leq\frac{1}{2}\left[\left(\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right)^2+\frac{1}{12}\right]\cdot(b-a)^2\|f''\|_\infty\\&\leq\frac{1}{6}(b-a)^2\|f''\|_\infty,\end{aligned}\tag{1.1}$$

for every $x\in[a,b]$.

This result is not the best possible. Namely Dedić et al. in [3] among other results obtained an improvement of the inequalities (1.1) valid also for a wider class of functions, as follows:

Theorem B. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is L-Lipschitzian function on $[a,b]$ i.e., $|f'(x)-f'(y)|\leq L(x-y)$, for all $x\in[a,b]$. Then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\&\leq\frac{1}{2}\left[\frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12}\right]\cdot(b-a)^2L\\&\leq\frac{1}{12}(b-a)^2L,\end{aligned}\tag{1.2}$$

for every $x\in[a,b]$, where

$$\delta(x):=\frac{\left|x-\frac{a+b}{2}\right|}{b-a},\ x\in[a,b].$$

A generalization due to Matić at al. of the result stated in Theorem A. can be found in [7]. We also refer reader to the recent article [4] by Dedić et al., in which an interesting further generalization of Ostrowski inequality was obtained.

In the recent paper [1] G.A. Anastassiou proved yet another result related to the one stated in Theorem A.:

Theorem C. *Let $f:[a,b]\to\mathbb{R}$ be 3-times differentiable on $[a,b]$. Assume that f''' is bounded on $[a,b]$. Then we obtain*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right.\\&\left.-\frac{f'(b)-f'(a)}{2(b-a)}\left[x^2-(a+b)x+\frac{a^2+b^2+4ab}{6}\right]\right|\\&\leq\|f'''\|_\infty\cdot\frac{A(x)}{(b-a)^3},\end{aligned}\tag{1.3}$$

where

$$A(x) = abx^4 - \frac{1}{3}a^2b^3x + \frac{1}{3}a^3bx^2 - ab^2x^3 - \frac{1}{3}a^3b^2x + \frac{1}{3}ab^3x^2 + a^2b^2x^2 - a^2bx^3 - \frac{1}{2}ax^5 - \frac{1}{2}bx^5 + \frac{1}{6}x^6 + \frac{3}{4}a^2x^4 + \frac{3}{4}b^2x^4 + \frac{1}{3}b^2a^4 - \frac{2}{3}a^3x^3 - \frac{2}{3}b^3x^3 - \frac{1}{3}b^3a^3 + \frac{5}{12}a^4x^2 + \frac{5}{12}b^4x^2 + \frac{1}{3}b^4a^2 - \frac{2}{15}ba^5 - \frac{2}{15}ab^5 - \frac{1}{6}a^5x - \frac{1}{6}b^5x + \frac{a^6}{20} + \frac{b^6}{20}.$$

Inequality is attained by

$$f(x) = (x-a)^3 + (b-x)^3,$$

in that case both sides of inequality equals zero.

We stated the inequality (1.3) with somewhat simplified expression within the absolute value signs at the left hand side.

Theorem C. was obtained by using the following generalized Montgomery identity also proved in [1]:

Theorem D. *Let $f : [a,b] \to \mathbb{R}$ be n-times differentiable on $[a,b]$, $n \in \mathbb{N}$. The n-th derivative $f^{(n)} : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Let $x \in [a,b]$. Define the kernel*

$$P(r,s) := \begin{cases} \frac{s-a}{b-a}, & a \le s \le r, \\ \frac{s-b}{b-a}, & r < s \le b, \end{cases}$$

where $r, s \in [a,b]$. Then it holds

$$\begin{aligned} & f(x) - \frac{1}{b-a}\int_a^b f(s_1)\,\mathrm{d}s_1 \\ & - \sum_{k=0}^{n-2} \frac{f^{(k)}(b) - f^{(k)}(a)}{b-a} \int_a^b \cdots \int_a^b P(x,s_1) \left(\prod_{i=1}^{k} P(s_i, s_{i+1}) \right) \mathrm{d}s_1 \cdots \mathrm{d}s_{k+1} \\ & = \int_a^b \cdots \int_a^b P(x,s_1) \left(\prod_{i=1}^{n-1} P(s_i, s_{i+1}) \right) f^{(n)}(s_n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n \end{aligned} \tag{1.4}$$

We make conventions that $\prod_{k=1}^{0} \bullet := 1$, $\sum_{k=0}^{-1} \bullet := 0$.

This paper is motivated mostly by Theorems C. and D. as well as with some interesting results from [3] and [4] which allow us to improve almost all results from [1]. First, in Section 2. we establish a connection between generalized Montgomery identity, Bernoulli polynomials and Euler identity. This connection enables us to improve the result stated in Theorem C. along with all other univariate Ostrowski type results from [1]. We do this in Section 3. Finally in Section 4. we give a generalization of one multivariate result from [1].

Remark 1. *It should be noted that recently a paper* [2] *appeared. The basic result in that paper is an identity for n-time differentiable functions which in turn is Montgomery identity* (1.4) *stated with somewhat different notation. Moreover, all the inequalities obtained in* [2] *follow from that identity and are obtained in quite the same manner as those in* [1]. *Hence, some of the results from our paper can be regarded as improvements of corresponding results from* [2].

2. Generalized Montgomery identity and Bernoulli polynomials

Consider the sequence $(B_k(t), k \geq 0)$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$B_k'(t) = kB_{k-1}(t), \quad k \geq 1, \quad B_0(t) = 1$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$

The values $B_k = B_k(0)$, $k \geq 0$ are known as Bernoulli numbers. For our purposes, the first five Bernoulli polynomials are

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},$$
$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\ B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}. \tag{2.1}$$

Let $(B_k^*(t), k \geq 0)$ be a sequence of periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \qquad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}.$$

¿From the properties of Bernoulli polynomials it easily follows that $B_0^*(t) = 1, B_1^*$ is discontinuous function with a jump of -1 at each integer, while B_k^* , $k \geq 2$, are continuous functions.

For every function $f : [a,b] \to \mathbb{R}$ with continuous n-th derivative, $n \geq 1$, and for every $x \in [a,b]$ the following formula is valid (see [6]):

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + T_{n-1}(x) + R_n(x)$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right], \tag{2.2}$$

with convention $T_0(x) = 0$, and

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t.$$

The formula (2.2) can be rewritten as

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \sum_{k=0}^{n-2} \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right)\frac{f^{(k)}(b) - f^{(k)}(a)}{b-a}$$
$$= \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t. \tag{2.3}$$

We claim that the formula (2.3) coincide with the generalized Montgomery identity stated in Theorem D. We prove this claim by the following two lemmas.

Lemma 1. *For all $k \in \{0,1,2,3,..\}$ we have*

$$\int_a^b \cdots \int_a^b P(x,s_1)\left(\prod_{i=1}^{k} P(s_i,s_{i+1})\right)\mathrm{d}s_1 \cdots \mathrm{d}s_{k+1}$$
$$= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right). \tag{2.4}$$

Proof. We prove our assertion by induction with respect to k. For $m = 1, 2, \cdots$ let us denote

$$Q_m(x) := \int_a^b \cdots \int_a^b P(x, s_1) P(s_1, s_2) \cdots P(s_{m-1}, s_m) \, \mathrm{d}s_1 \cdots \mathrm{d}s_m.$$

For $k = 0$ the left hand side of (2.4) is equal to

$$\begin{aligned} Q_1(x) = \int_a^b P(x, s_1) \, \mathrm{d}s_1 &= \int_a^x \frac{s_1 - a}{b-a} \mathrm{d}s_1 + \int_x^b \frac{s_1 - b}{b-a} \mathrm{d}s_1 \\ &= \frac{1}{2(b-a)} \left[(x-a)^2 - (x-b)^2 \right] = x - \frac{a+b}{2}, \end{aligned}$$

while the right hand side of (2.4) is equal to

$$\frac{(b-a)^1}{1!} B_1\left(\frac{x-a}{b-a}\right) = (b-a)\left(\frac{x-a}{b-a} - \frac{1}{2}\right) = x - \frac{a+b}{2}.$$

Hence the formula (2.4) is valid for $k = 0$. Now suppose that for some $k > 0$

$$Q_k(y) = \frac{(b-a)^k}{k!} B_k\left(\frac{y-a}{b-a}\right)$$

is true for all $y \in [a, b]$. Then we have

$$\begin{aligned} Q_{k+1}(x) &= \int_a^b P(x, s_1) Q_k(s_1) \, \mathrm{d}s_1 = \int_a^b P(x, s_1) \frac{(b-a)^k}{k!} B_k\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 \\ &= \frac{(b-a)^k}{k!} \left[\int_a^x \frac{s_1 - a}{b-a} B_k\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 + \int_x^b \frac{s_1 - b}{b-a} B_k\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 \right]. \end{aligned}$$

Because of the properties of Bernoulli polynomials we have

$$\frac{\mathrm{d}}{\mathrm{d}s_1} \left[\frac{b-a}{m} B_m\left(\frac{s_1 - a}{b-a}\right) \right] = B_{m-1}\left(\frac{s_1 - a}{b-a}\right), \ m \geq 1,$$

and partial integration yields

$$\begin{aligned} Q_{k+1}(x) &= \frac{(b-a)^k}{k!} \left[\frac{s_1 - a}{k+1} \, B_{k+1}\left(\frac{s_1 - a}{b-a}\right) \Big|_a^x - \int_a^x \frac{1}{k+1} B_{k+1}\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 \right] \\ &\quad + \frac{(b-a)^k}{k!} \left[\frac{s_1 - b}{k+1} \, B_{k+1}\left(\frac{s_1 - a}{b-a}\right) \Big|_x^b - \int_x^b \frac{1}{k+1} B_{k+1}\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 \right] \\ &= \frac{(b-a)^k}{k!} \left[\frac{x-a}{k+1} B_{k+1}\left(\frac{x-a}{b-a}\right) - \frac{x-b}{k+1} B_{k+1}\left(\frac{x-a}{b-a}\right) \right] \\ &\quad - \frac{(b-a)^k}{(k+1)!} \int_a^b B_{k+1}\left(\frac{s_1 - a}{b-a}\right) \mathrm{d}s_1 \\ &= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right) - \frac{(b-a)^k}{(k+1)!} \cdot \frac{b-a}{k+2} \cdot \, B_{k+2}\left(\frac{s_1 - a}{b-a}\right) \Big|_a^b \\ &= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right) - \frac{(b-a)^{k+1}}{(k+2)!} \left[B_{k+2}(1) - B_{k+2}(0) \right] \\ &= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1}\left(\frac{x-a}{b-a}\right), \end{aligned}$$

where we used the fact that $B_m(1) = B_m(0)$ for all $m \geq 2$. We see that (2.4) is valid for $k+1$ and our assertion is proved. □

Lemma 2. *For all $n \in \mathbb{N}$ we have*
a)

$$\int_a^b \cdots \int_a^b P(x,s_1)\left(\prod_{i=1}^{n-1} P(s_i,s_{i+1})\right) f^{(n)}(s_n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n$$
$$= \frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-t}{b-a}\right)\right] f^{(n)}(t)\,\mathrm{d}t.$$

b)

$$\int_a^b \cdots \int_a^b P(x,s_1)\left(\prod_{i=1}^{n-1} P(s_i,s_{i+1})\right) \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1}$$
$$= \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-s_n}{b-a}\right)\right].$$

Here we make convention that $\prod_{i=1}^{0} \bullet := 1$.

Proof. It is obvious that **a)** follows from **b)** so that we only have to prove **b)**. We do this again by induction. Let us denote

$$q_k(x,t) = \int_a^b \cdots \int_a^b P(x,s_1)\,P(s_1,s_2)\cdots P(s_k,t)\,\mathrm{d}s_1 \cdots \mathrm{d}s_k. \tag{2.5}$$

Then our claim is that

$$q_k(x,t) = \frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{x-a}{b-a}\right) - B_{k+1}^*\left(\frac{x-t}{b-a}\right)\right] \tag{2.6}$$

is true for all $k \in \{0,1,2,3,...\}$. For $k=0$ we have

$$q_0(x,t) = P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x, \\ \frac{t-b}{b-a}, & x < t \leq b, \end{cases}$$

while

$$\frac{1}{1!}\left[B_1\left(\frac{x-a}{b-a}\right) - B_1^*\left(\frac{x-t}{b-a}\right)\right] = \begin{cases} \left(\frac{x-a}{b-a} - \frac{1}{2}\right) - \left(\frac{x-t}{b-a} - \frac{1}{2}\right), & a \leq t \leq x \\ \left(\frac{x-a}{b-a} - \frac{1}{2}\right) - \left(\frac{x-t}{b-a} + 1 - \frac{1}{2}\right), & x < t \leq b \end{cases}$$
$$= \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a,}, & x < t \leq b \end{cases}$$

and it is clear that our assertion is true for $k=0$. Further suppose that (2.6) is true for some $k \geq 0$ and note that in that case we can write

$$q_{k+1}(x,t) = \int_a^b \cdots \int_a^b P(x,s_1)\,P(s_1,s_2)\cdots P(s_{k+1},t)\,\mathrm{d}s_1 \cdots \mathrm{d}s_{k+1.}$$
$$= \int_a^b P(x,s_1)\,q_k(s_1,t)\,\mathrm{d}s_1$$
$$= \int_a^b P(x,s_1)\,\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}^*\left(\frac{s_1-t}{b-a}\right)\right]\mathrm{d}s_1.$$

Now we must consider two cases: $t \leq x$ and $t > x$. In the first case when $t \leq x$ we have

$$\begin{aligned} q_{k+1}(x,t) = &\int_a^t \frac{s_1-a}{b-a} \cdot \frac{(b-a)^k}{(k+1)!} \left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}\left(\frac{s_1-t}{b-a}+1\right)\right] \mathrm{d}s_1 \\ &+ \int_t^x \frac{s_1-a}{b-a} \cdot \frac{(b-a)^k}{(k+1)!} \left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}\left(\frac{s_1-t}{b-a}\right)\right] \mathrm{d}s_1 \\ &+ \int_x^b \frac{s_1-b}{b-a} \cdot \frac{(b-a)^k}{(k+1)!} \left[B_{k+1}\left(\frac{s_1-a}{b-a}\right) - B_{k+1}\left(\frac{s_1-t}{b-a}\right)\right] \mathrm{d}s_1. \end{aligned}$$

Let us denote the three integrals at the right hand side respectively by I_1, I_2 and I_3. Using partial integration and properties of Bernoulli polynomials we get

$$\begin{aligned} I_1 = &\frac{(b-a)^k}{(k+2)!}(s_1-a)\left[B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}+1\right)\right]\Bigg|_a^t \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}\left(\frac{s_1-a}{b-a}\right) - B_{k+3}\left(\frac{s_1-t}{b-a}+1\right)\right]\Bigg|_a^t \\ = &\frac{(b-a)^k}{(k+2)!}(t-a)\left[B_{k+2}\left(\frac{t-a}{b-a}\right) - B_{k+2}(1)\right] \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}\left(\frac{t-a}{b-a}\right) - B_{k+2}(0) - B_{k+2}(1) + B_{k+3}\left(\frac{b-t}{b-a}\right)\right], \end{aligned}$$

$$\begin{aligned} I_2 = &\frac{(b-a)^k}{(k+2)!}(s_1-a)\left[B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}\right)\right]\Bigg|_t^x \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}\left(\frac{s_1-a}{b-a}\right) - B_{k+3}\left(\frac{s_1-t}{b-a}\right)\right]\Bigg|_t^x \\ = &\frac{(b-a)^k}{(k+2)!}(x-a)\left[B_{k+2}\left(\frac{x-a}{b-a}\right) - B_{k+2}\left(\frac{x-t}{b-a}\right)\right] \\ &- \frac{(b-a)^k}{(k+2)!}(t-a)\left[B_{k+2}\left(\frac{t-a}{b-a}\right) - B_{k+2}(0)\right] \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}\left(\frac{x-a}{b-a}\right) - B_{k+3}\left(\frac{t-a}{b-a}\right) - B_{k+3}\left(\frac{x-t}{b-a}\right) + B_{k+3}(0)\right] \end{aligned}$$

and

$$\begin{aligned} I_3 = &\frac{(b-a)^k}{(k+2)!}(s_1-b)\left[B_{k+2}\left(\frac{s_1-a}{b-a}\right) - B_{k+2}\left(\frac{s_1-t}{b-a}\right)\right]\Bigg|_x^b \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}\left(\frac{s_1-a}{b-a}\right) - B_{k+3}\left(\frac{s_1-t}{b-a}\right)\right]\Bigg|_x^b \\ = &-\frac{(b-a)^k}{(k+2)!}(x-b)\left[B_{k+2}\left(\frac{x-a}{b-a}\right) - B_{k+2}\left(\frac{x-t}{b-a}\right)\right] \\ &- \frac{(b-a)^{k+1}}{(k+3)!}\left[B_{k+3}(1) - B_{k+3}\left(\frac{x-a}{b-a}\right) - B_{k+3}\left(\frac{b-t}{b-a}\right) + B_{k+3}\left(\frac{x-t}{b-a}\right)\right]. \end{aligned}$$

Finally, after adding and simplifying we obtain

$$\begin{aligned} q_{k+1}(x,t) &= I_1 + I_2 + I_3 \\ &= \frac{(b-a)^k}{(k+2)!}\left[(x-a)-(x-b)\right]\left[B_{k+2}\left(\frac{x-a}{b-a}\right) - B_{k+2}\left(\frac{x-t}{b-a}\right)\right] \\ &= \frac{(b-a)^{k+1}}{(k+2)!}\left[B_{k+2}\left(\frac{x-a}{b-a}\right) - B^*_{k+2}\left(\frac{x-t}{b-a}\right)\right], \end{aligned}$$

which is desired result for the case when $t \le x$. The case when $t > x$ is handled quite analogously and we get the same formula again. So, the formula (2.6) is valid with k replaced with $k+1$, which proves our assertion. □

3. Main results

As we noted in Introduction, Theorem C. in [1] was obtained as one possible generalization of the result stated in Theorem A. The proof of that result in [1] was carried out via generalized Montgomery identity (1.4) in the following way. For $n = 3$ the identity (1.4) is just

$$\begin{aligned} & f(x) - \frac{1}{b-a}\int_a^b f(s_1)\,\mathrm{d}s_1 - \frac{f(b)-f(a)}{b-a}\int_a^b P(x,s_1)\,\mathrm{d}s_1 \\ & - \frac{f'(b)-f'(a)}{b-a}\int_a^b\int_a^b P(x,s_1)\,P(s_1,s_2)\,\mathrm{d}s_1\mathrm{d}s_2 \\ & = \int_a^b\int_a^b\int_a^b P(x,s_1)\,P(s_1,s_2)\,P(s_2,s_3)\,f'''(s_3)\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3, \end{aligned}$$

and the inequality (1.3) was obtained after calculating all the integrals in the following inequality

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(s_1)\,\mathrm{d}s_1 - \frac{f(b)-f(a)}{b-a}\int_a^b P(x,s_1)\,\mathrm{d}s_1 \right. \\ & \left. - \frac{f'(b)-f'(a)}{b-a}\int_a^b\int_a^b P(x,s_1)\,P(s_1,s_2)\,\mathrm{d}s_1\mathrm{d}s_2 \right| \\ & \le \|f'''\|_\infty \int_a^b |P(x,s_1)| \left(\int_a^b |P(s_1,s_2)| \left(\int_a^b |P(s_2,s_3)|\,\mathrm{d}s_3\right)\mathrm{d}s_2\right)\mathrm{d}s_1. \end{aligned}$$

Similarly, if we do the same for $n = 2$, then by calculating all the integrals in the inequality

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(s_1)\,\mathrm{d}s_1 - \frac{f(b)-f(a)}{b-a}\int_a^b P(x,s_1)\,\mathrm{d}s_1 \right| \\ & \le \|f''\|_\infty \int_a^b |P(x,s_1)| \left(\int_a^b |P(s_1,s_2)|\,\mathrm{d}s_2\right)\mathrm{d}s_1 \end{aligned}$$

we get the inequality (1.1).

Instead of the approach explained above, we suggest the better one in which the key role plays the connection between two identities, (1.4) and (2.3), established in the preceding section.

Let us denote the left hand side in (1.4), that is the left hand side in (2.3), by $R_n(x)$. Then by Lemma 2. we have

$$\begin{aligned} R_n(x) &= \int_a^b \cdots \int_a^b P(x,s_1) P(s_1,s_2) \cdots P(s_{n-1},s_n) f^{(n)}(s_n) \,\mathrm{d}s_1 \cdots \mathrm{d}s_n \\ &= \int_a^b q_{n-1}(x,s_n) f^{(n)}(s_n) \,\mathrm{d}s_n, \end{aligned}$$

where $q_{n-1}(\cdot,\cdot)$ is defined by (2.5). Now, for any function f satisfying the assumptions of Theorem D. the following inequality obviously holds

$$|R_n(x)| \le \left\| f^{(n)} \right\|_\infty \int_a^b |q_{n-1}(x,s_n)| \,\mathrm{d}s_n, \tag{3.1}$$

for all $x \in [a,b]$. The crucial point in this approach is that we can exactly evaluate the integral $\int_a^b |q_{n-1}(x,s_n)| \,\mathrm{d}s_n$, using the formula (2.6).

For example, if we do this for $n=2$ we get

$$\begin{aligned} \int_a^b |q_1(x,t)| \,\mathrm{d}t &= \frac{b-a}{2} \int_a^b \left| B_2^*\left(\frac{x-t}{b-a}\right) - B_2\left(\frac{x-a}{b-a}\right) \right| \mathrm{d}t \\ &= \frac{(b-a)^2}{2} \int_0^1 \left| B_2(t) - B_2\left(\frac{x-a}{b-a}\right) \right| \mathrm{d}t \\ &= \frac{(b-a)^2}{2} \int_0^1 \left| t^2 - t - \left(\frac{x-a}{b-a}\right)^2 + \frac{x-a}{b-a} \right| \mathrm{d}t \\ &= \frac{1}{2}\left[\frac{8}{3}\delta^3(x) - \delta^2(x) + \frac{1}{12}\right] \cdot (b-a)^2, \end{aligned}$$

where $\delta(x) := \frac{\left|x - \frac{a+b}{2}\right|}{b-a}$, $x \in [a,b]$. On the other side using Lemma 1. and (2.1) we easily get

$$R_2(x) = f(x) - \frac{1}{b-a}\int_a^b f(t) \,\mathrm{d}t - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right),$$

so that the inequality (3.1) reduces to the result from Theorem B. for the special case when $f : [a,b] \to \mathbb{R}$ is twice differentiable with bounded second derivative on $[a,b]$ and $L = \|f''\|_\infty$.

Now we come to the main result, that is improvement of the inequality (1.3).

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be 3-times differentiable on $[a,b]$. Assume that f''' is bounded on $[a,b]$. Then, for all $x \in [a,b]$ we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t) \,\mathrm{d}t - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right. \\ &\left. - \frac{f'(b)-f'(a)}{2(b-a)}\left[x^2 - (a+b)x + \frac{a^2+b^2+4ab}{6}\right] \right| \\ &\le \|f'''\|_\infty \cdot \frac{(b-a)^3}{6} I\left(\frac{x-a}{b-a}\right), \end{aligned} \tag{3.2}$$

where

$$I(\lambda)=\begin{cases}-\frac{3}{2}(t_1)^4+2(t_1)^3-\frac{1}{2}(t_1)^2+\frac{3}{2}\lambda^4-\lambda^3-\lambda^2+\frac{1}{2}\lambda, & \lambda\in[0,\frac{3-\sqrt{3}}{6}]\\ \frac{3}{2}(t_1)^4-2(t_1)^3+\frac{1}{2}(t_1)^2-\frac{3}{2}\lambda^4+3\lambda^3-2\lambda^2+\frac{1}{2}\lambda, & \lambda\in(\frac{3-\sqrt{3}}{6},\frac{1}{2}]\\ \frac{3}{2}(t_2)^4-2(t_2)^3+\frac{1}{2}(t_2)^2-\frac{3}{2}\lambda^4+\lambda^3+\lambda^2-\frac{1}{2}\lambda, & \lambda\in(\frac{1}{2},\frac{3+\sqrt{3}}{6}]\\ -\frac{3}{2}(t_2)^4+2(t_2)^3-\frac{1}{2}(t_2)^2+\frac{3}{2}\lambda^4-3\lambda^3+2\lambda^2-\frac{1}{2}\lambda, & \lambda\in(\frac{3+\sqrt{3}}{6},1]\end{cases}$$

and

$$t_1=\frac{3}{4}-\frac{1}{2}\lambda-\frac{1}{2}\sqrt{\frac{1}{4}+3\lambda-3\lambda^2},\ t_2=\frac{3}{4}-\frac{1}{2}\lambda+\frac{1}{2}\sqrt{\frac{1}{4}+3\lambda-3\lambda^2}. \tag{3.3}$$

Furthermore, for the term $A(x)/(b-a)^3$ *in* (1.3) *we have*

$$\frac{A(x)}{(b-a)^3}=\frac{(b-a)^3}{6}B\left(\frac{x-a}{b-a}\right), \tag{3.4}$$

where

$$B(\lambda)=\lambda^6-3\lambda^5+\frac{9}{2}\lambda^4-4\lambda^3+\frac{5}{2}\lambda^2-\lambda+\frac{3}{10},$$

and for all $\lambda\in[0,1]$ *the following inequalities are valid*

$$\frac{1}{32}\le I(\lambda)\le\frac{\sqrt{3}}{36}<\frac{41}{320}\le B(\lambda)\le\frac{3}{10}. \tag{3.5}$$

Proof. For given $x\in[a,b]$ let us define $\lambda=\frac{x-a}{b-a}$ and note that $0\le\lambda\le1$. Now, for $n=3$, applying formula (2.6) and using (2.1) we get

$$\begin{aligned}\int_a^b|q_2(x,t)|\,\mathrm{d}t&=\frac{(b-a)^2}{6}\int_a^b\left|B_3^*\left(\frac{x-t}{b-a}\right)-B_3\left(\frac{x-a}{b-a}\right)\right|\mathrm{d}t\\&=\frac{(b-a)^3}{6}\int_0^1|B_3(t)-B_3(\lambda)|\,\mathrm{d}t\\&=\frac{(b-a)^3}{6}\int_0^1\left|t^3-\frac{3}{2}t^2+\frac{1}{2}t-\lambda^3+\frac{3}{2}\lambda^2-\frac{1}{2}\lambda\right|\mathrm{d}t.\end{aligned}$$

The polynomial

$$p(t)=B_3(t)-B_3(\lambda)=t^3-\frac{3}{2}t^2+\frac{1}{2}t-\lambda^3+\frac{3}{2}\lambda^2-\frac{1}{2}\lambda$$

has three roots. One of them obviously is $t_0=\lambda$, and two others are t_1 and t_2 given by (3.3). For the sake of simplicity let us denote the integral $\int_0^1|p(t)|\,\mathrm{d}t$ by $\tilde{I}(\lambda)$. It is easily seen that we must consider four different cases:
(i) if $\lambda\in\left(0,\frac{3-\sqrt{3}}{6}\right)$, then $0<\lambda<t_1<1<t_2$ and we get

$$\tilde{I}(\lambda)=(2\lambda-2t_1+1)B_3(\lambda)+\frac{1}{2}[B_4(t_1)-B_4(\lambda)];$$

(ii) if $\lambda\in\left(\frac{3-\sqrt{3}}{6},\frac{1}{2}\right)$, then $0<t_1<\lambda<1<t_2$ and we get

$$\tilde{I}(\lambda)=(2t_1-2\lambda+1)B_3(\lambda)+\frac{1}{2}[B_4(\lambda)-B_4(t_1)];$$

(iii) if $\lambda\in\left(\frac{1}{2},\frac{3+\sqrt{3}}{6}\right)$, then $t_1<0<\lambda<t_2<1$ and we get

$$\tilde{I}(\lambda)=(2t_2-2\lambda-1)B_3(\lambda)+\frac{1}{2}[B_4(\lambda)-B_4(t_2)];$$

(iv) if $\lambda \in \left(\frac{3+\sqrt{3}}{6}, 1\right)$, then $t_1 < 0 < t_2 < \lambda < 1$ and we get

$$\tilde{I}(\lambda) = (2\lambda - 2t_2 - 1)B_3(\lambda) + \frac{1}{2}[B_4(t_2) - B_4(\lambda)].$$

Now, using (2.1) and the fact that t_1 and t_2 are the roots of $p(t)$ which implies

$$(t_1)B_3(\lambda) = (t_1)^4 - \frac{3}{2}(t_1)^3 + \frac{1}{2}(t_1)^2, \quad (t_2)B_3(\lambda) = (t_2)^4 - \frac{3}{2}(t_2)^3 + \frac{1}{2}(t_2)^2,$$

we easily see that $\tilde{I}(\lambda)$ coincides with $I(\lambda)$ as stated in Theorem. Hence, the first assertion in Theorem is proved. To check out the formula (3.4) we set $x = a+\lambda(b-a)$ in the expression for $A(x)$ and then simplify it, or simply force Mathematica 4 to do this job for us. Finally, the inequalities stated in (3.5) are consequences of the fact that

$$\min_{\lambda\in[0,1]} I(\lambda) = I(0) = I\left(\frac{1}{2}\right) = I(1) = \frac{1}{32},$$
$$\max_{\lambda\in[0,1]} I(\lambda) = I\left(\frac{3-\sqrt{3}}{6}\right) = I\left(\frac{3+\sqrt{3}}{6}\right) = \frac{\sqrt{3}}{36}$$

and

$$\min_{\lambda\in[0,1]} B(\lambda) = B\left(\frac{1}{2}\right) = \frac{41}{320}, \quad \max_{\lambda\in[0,1]} B(\lambda) = B(0) = B(1) = \frac{3}{10}$$

□

¿From Theorem 1. it is evident that our approach gives strictly better estimates than those of Anastassiou. Also, the best estimates are obtained for $\lambda \in \left\{0, \frac{1}{2}, 1\right\}$, i.e. in corresponding trapezoid and midpoint inequalities:

Corollary 1. *Under the assumptions of Theorem 1. we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{b-a}{12}[f'(b) - f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \|f'''\|_\infty \cdot \frac{(b-a)^3}{192} \tag{3.6}$$

and

$$\left|f\left(\frac{a+b}{2}\right) + \frac{b-a}{24}[f'(b) - f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|$$
$$\leq \|f'''\|_\infty \cdot \frac{(b-a)^3}{192}. \tag{3.7}$$

Proof. For $x = a$, $\lambda = 0$ or $x = b$, $\lambda = 1$ the inequality (3.2) reduces to the trapezoid inequality (3.6). Similarly for $x = \frac{a+b}{2}$ the inequality (3.2) reduces to the midpoint inequality (3.7). □

We give yet another interesting special case which improves the corresponding Anastassiou's result [1, Corollary 4.]:

Corollary 2. *Suppose all the assumptions of Theorem 1. hold. Additionally assume that* $f(a) = f(b)$ *and* $f'(a) = f'(b)$. *Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \leq \left\| f''' \right\|_\infty \cdot \frac{(b-a)^3}{6} I\left(\frac{x-a}{b-a}\right).$$

Proof. Obvious. □

Next we proceed with exploiting our idea to improve the rest of univariate results from [1]. Those results are consequences of generalized Montgomery's identity [1, Theorem 2.]. First we state that identity with somewhat changed notation:

Proposition 1. *Let* $f : [a,b] \to \mathbb{R}$ *be n-times differentiable on* $[a,b], n \in \mathbb{N}$. *Assume n-th derivative* $f^{(n)} : [a,b] \to \mathbb{R}$ *to be integrable on* $[a,b]$. *Let also* $g : [a,b] \to \mathbb{R}$ *be a function of bounded variation , such that* $g(a) \neq g(b)$. *For any* $x \in [a,b]$ *define*

$$P_g(x,t) := \begin{cases} \frac{g(t)-g(a)}{g(b)-g(a)}, & a \leq t \leq x, \\ \frac{g(t)-g(b)}{g(b)-g(a)}, & x < t \leq b. \end{cases}$$

Then it holds

$$\begin{aligned} & f(x) - \frac{\int_a^b f(s_1)\,\mathrm{d}g(s_1)}{g(b)-g(a)} \\ & - \sum_{k=0}^{n-2} \frac{\int_a^b f^{(k+1)}(s_1)\,\mathrm{d}g(s_1)}{g(b)-g(a)} \int_a^b \cdots \int_a^b P_g(x,s_1) \left(\prod_{i=1}^{k} P_g(s_i, s_{i+1}) \right) \mathrm{d}s_1 \cdots \mathrm{d}s_{k+1} \\ & = \int_a^b \cdots \int_a^b P_g(x,s_1) \left(\prod_{i=1}^{n-1} P_g(s_i, s_{i+1}) \right) f^{(n)}(s_n)\,\mathrm{d}s_1 \cdots \mathrm{d}s_n. \end{aligned} \tag{3.8}$$

Proof. See [1, Theorem 2.]. □

Theorem 2. *Suppose all the assumptions of Proposition 1. are satisfied. Additionally assume that* $\left\| f^{(n)} \right\|_\infty < +\infty$. *If* $\tilde{R}_{g,n}(x)$ *is the left hand side of* (3.8), *then*

$$\left| \tilde{R}_{g,n}(x) \right| \leq \left\| f^{(n)} \right\|_\infty \cdot \int_a^b \left| \tilde{q}_{g,n-1}(x,s_n) \right| \mathrm{d}s_n, \tag{3.9}$$

where

$$\tilde{q}_{g,k}(x,t) = \int_a^b \cdots \int_a^b P_g(x,s_1) P_g(s_1,s_2) \cdots P_g(s_k,t)\,\mathrm{d}s_1 \cdots \mathrm{d}s_k.$$

Proof. The identity (3.8) can be rewritten as

$$\tilde{R}_{g,n}(x) = \int_a^b \tilde{q}_{g,n-1}(x,s_n) f^{(n)}(s_n)\,\mathrm{d}s_n \tag{3.10}$$

and (3.9) follows immediately, since

$$\left| \int_a^b \tilde{q}_{g,n-1}(x,s_n) f^{(n)}(s_n)\,\mathrm{d}s_n \right| \leq \left\| f^{(n)} \right\|_\infty \cdot \int_a^b \left| \tilde{q}_{g,n-1}(x,s_n) \right| \mathrm{d}s_n,$$

by triangle inequality □

Remark 2. *The inequality* (3.9) *is an improvement of the corresponding result* [1, Theorem 8.] *where*

$$\int_a^b \cdots \int_a^b \left|P_g\left(x,s_1\right)\right| \left(\prod_{i=1}^{n-1} \left|P_g\left(s_i,s_{i+1}\right)\right|\right) \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

stands in place of $\int_a^b \left|\tilde{q}_{g,n-1}\left(x,s_n\right)\right| \mathrm{d}s_n$

In the special case when $g\left(x\right)=x$ *we have*

$$\tilde{q}_{g,k}\left(x,t\right)=q_k\left(x,t\right)=\frac{\left(b-a\right)^k}{\left(k+1\right)!}\left[B_{k+1}\left(\frac{x-a}{b-a}\right)-B_{k+1}^*\left(\frac{x-t}{b-a}\right)\right]$$

and (3.9) *reduces to the result from* [3, Theorem 7.]

For the sake of completeness we also present L_p Ostrowski type result:

Theorem 3. *Suppose all the assumptions of Proposition 1. are satisfied. Additionally assume that* $\left\|f^{(n)}\right\|_p<+\infty$ *for some* $p\geq 1$. *If* $\tilde{R}_{g,n}(x)$ *is the left hand side of* (3.8) *and* $r\leq+\infty$ *such that* $\frac{1}{p}+\frac{1}{r}=1$, *then*

$$\left|\tilde{R}_{g,n}(x)\right|\leq\left\|f^{(n)}\right\|_p\cdot\left\|\tilde{q}_{g,n-1}\left(x,\bullet\right)\right\|_r.$$

Proof. The result follows directly from (3.10) by the Hölder inequality. □

Remark 3. *The above result improves the corresponding* L_p *Ostrowski type results from* [1]. *Also, in the special case when* $g\left(x\right)=x$, *the above Theorem recaptures the results from* [3, Theorem 9., Corollary 3.].

4. Generalization of the multivariate result

In this section we consider one multivariate result from [1]. We give it here with a slight changed notation (see [1, Theorem 3.]):

Proposition 2. *Let* Q *be a compact convex subset of* $\mathbb{R}^n$, $n\geq 2$; $\mathbf{x}=\left(x_1,...,x_n\right)\in Q$ *and* $\mathbf{0}=\left(0,...,0\right)\in Q$. *Let* $f\in C^2\left(Q\right)$ *and assume that all partial derivatives of* f *of order one are coordinatewise absolutely continuous functions. Then*

$$\begin{aligned} f\left(\mathbf{x}\right)=&\int_0^1 f\left(t_1\mathbf{x}\right)\mathrm{d}t_1+\sum_{i=1}^n x_i\int_0^1\int_0^1 t_1\frac{\partial f\left(t_1t_2\mathbf{x}\right)}{\partial x_i}\mathrm{d}t_1\mathrm{d}t_2\\ &+\sum_{i=1}^n\sum_{j=1}^n x_ix_j\int_0^1\int_0^1 t_1^2t_2\frac{\partial^2 f\left(t_1t_2\mathbf{x}\right)}{\partial x_i\partial x_j}\mathrm{d}t_1\mathrm{d}t_2. \end{aligned} \tag{4.1}$$

Now, if we assume that $f\in C^3\left(Q\right)$ and that all partial derivatives of f of order one and two are coordinatewise absolutely continuous functions, then we can apply the identity (see [1])

$$f\left(\mathbf{x}\right)=\int_0^1 f\left(t_1\mathbf{x}\right)\mathrm{d}t_1+\sum_{i=1}^n x_i\int_0^1 t_1\frac{\partial f\left(t_1\mathbf{x}\right)}{\partial x_i}\mathrm{d}t_1$$

to the function $\frac{\partial^2 f}{\partial x_i\partial x_j}$. Doing so we get

$$\frac{\partial^2 f\left(\mathbf{x}\right)}{\partial x_i\partial x_j}=\int_0^1\frac{\partial^2 f\left(t_3\mathbf{x}\right)}{\partial x_i\partial x_j}\mathrm{d}t_3+\sum_{k=1}^n x_k\int_0^1 t_3\frac{\partial^3 f\left(t_3\mathbf{x}\right)}{\partial x_i\partial x_j\partial x_k}\mathrm{d}t_3,$$

so that

$$t_1^2 t_2 \frac{\partial^2 f(t_1 t_2 \mathbf{x})}{\partial x_i \partial x_j} = t_1^2 t_2 \int_0^1 \frac{\partial^2 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j} \mathrm{d}t_3 + \sum_{k=1}^{n} t_1^3 t_2^2 x_k \int_0^1 t_3 \frac{\partial^3 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j \partial x_k} \mathrm{d}t_3,$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 t_1^2 t_2 \frac{\partial^2 f(t_1 t_2 \mathbf{x})}{\partial x_i \partial x_j} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \int_0^1 \int_0^1 t_1^2 t_2 \int_0^1 \frac{\partial^2 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \\ &+ \int_0^1 \int_0^1 \sum_{k=1}^{n} t_1^3 t_2^2 x_k \int_0^1 t_3 \frac{\partial^3 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j \partial x_k} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3. \end{aligned}$$

Finally we obtain

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \int_0^1 \int_0^1 t_1^2 t_2 \frac{\partial^2 f(t_1 t_2 \mathbf{x})}{\partial x_i \partial x_j} \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \int_0^1 \int_0^1 t_1^2 t_2 \int_0^1 \frac{\partial^2 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \int_0^1 \int_0^1 \sum_{k=1}^{n} x_k t_1^3 t_2^2 \int_0^1 t_3 \frac{\partial^3 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j \partial x_k} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3. \end{aligned}$$

Putting this in the identity (4.1) we obtain new identity

$$\begin{aligned} f(\mathbf{x}) = &\int_0^1 f(t_1 \mathbf{x}) \, \mathrm{d}t_1 + \sum_{i=1}^{n} x_i \int_0^1 \int_0^1 t_1 \frac{\partial f(t_1 t_2 \mathbf{x})}{\partial x_i} \mathrm{d}t_1 \mathrm{d}t_2 \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \int_0^1 \int_0^1 \int_0^1 t_1^2 t_2 \frac{\partial^2 f(t_1 t_2 t_3 \mathrm{x})}{\partial x_i \partial x_j} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3 \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} x_k x_i x_j \int_0^1 \int_0^1 \int_0^1 t_1^3 t_2^2 t_3 \frac{\partial^3 f(t_1 t_2 t_3 \mathbf{x})}{\partial x_i \partial x_j \partial x_k} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}t_3. \end{aligned}$$

Proceeding in this way we easily obtain the following generalization of the identity (4.1):

Theorem 4. *Let Q be a compact convex subset of $\mathbb{R}^n$, $n \geq 2$, $\mathbf{x} = (x_1, ..., x_n) \in Q$ and $\mathbf{0} = (0, ..., 0) \in Q$. Let $f \in C^m(Q)$ and assume that all partial derivatives of f of order less than or equal to $m-1$ are coordinatewise absolutely continuous*

functions. Then

$$
\begin{aligned}
&f(\mathbf{x}) \\
&= \int_0^1 f(t_1\mathbf{x})\,\mathrm{d}t_1 + \sum_{i=1}^{n} x_i \int_0^1 \int_0^1 t_1 \frac{\partial f(t_1t_2\mathbf{x})}{\partial x_i}\mathrm{d}t_1\mathrm{d}t_2 \\
&+ \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} x_{i_1}x_{i_2} \int_0^1 \int_0^1 \int_0^1 t_1^2 t_2 \frac{\partial^2 f(t_1t_2t_3\mathbf{x})}{\partial x_{i_1}\partial x_{i_2}}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \\
&+ \cdots \\
&+ \sum_{i_1=1}^{n}\cdots\sum_{i_{m-1}=1}^{n} x_{i_1}\cdots x_{i_{m-1}} \int_0^1 \cdots \int_0^1 t_1^{m-1}\cdots t_{m-1} \frac{\partial^{m-1} f(t_1\cdots t_m\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_{m-1}}}\mathrm{d}t_1\cdots\mathrm{d}t_m \\
&+ \sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} x_{i_1}\cdots x_{i_m} \int_0^1 \cdots \int_0^1 t_1^m t_2^{m-1}\cdots t_m \frac{\partial^m f(t_1t_2\cdots t_m\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_m}}\mathrm{d}t_1\cdots\mathrm{d}t_m.
\end{aligned}
$$

Proof. Similar as for the above explained case $m = 3$. □

Theorem 5. *Suppose that all the assumptions of Theorem 4 hold. Additionally assume that*

$$\gamma_{i_1,..,i_m} := \left\| \frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}} \right\|_\infty < +\infty,$$

for all $i_1,..,i_m \in \{1,2,\cdots,n\}$. *If*

$$
\begin{aligned}
&R_m(\mathbf{x}) \\
&= f(\mathbf{x}) - \int_0^1 f(t_1\mathbf{x})\,\mathrm{d}t - \sum_{i=1}^{n} x_i \int_0^1 \int_0^1 t_1 \frac{\partial f(t_1t_2\mathbf{x})}{\partial x_i}\mathrm{d}t_1\mathrm{d}t_2 \\
&- \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} x_{i_1}x_{i_2} \int_0^1 \int_0^1 \int_0^1 t_1^2 t_2 \frac{\partial^2 f(t_1t_2t_3\mathbf{x})}{\partial x_{i_1}\partial x_{i_2}}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3 \\
&- \cdots \\
&- \sum_{i_1=1}^{n}\cdots\sum_{i_{m-1}=1}^{n} x_{i_1}\cdots x_{i_{m-1}} \int_0^1 \cdots \int_0^1 t_1^{m-1}\cdots t_{m-1} \frac{\partial^{m-1} f(t_1\cdots t_m\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_{m-1}}}\mathrm{d}t_1\cdots\mathrm{d}t_m,
\end{aligned}
$$

then

$$|R_m(\mathbf{x})| \le \frac{1}{(m+1)!}\left(\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} |x_{i_1}|\cdots|x_{i_m}|\cdot\gamma_{i_1,..,i_m}\right).$$

Proof. Follows from the identity stated in Theorem 4. and the fact that

$$\int_0^1 \cdots \int_0^1 t_1^m t_2^{m-1}\cdots t_m \mathrm{d}t_1\cdots\mathrm{d}t_m = \frac{1}{(m+1)!}.$$

□

Remark 4. *In the case when* $m = 2$ *the inequality from Theorem 5. reduces to Anastasiou's result* [1, Theorem 9.].

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An R-order four iteration in Banach space

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Abstract

In this paper, we apply a R-order four iterative method to solve non-linear operator equations in Banach spaces. We prove a semilocal convergence theorem which guarantees local convergence with R-order four under conditions similar to those of the Newton-Kantorovich theorem, assuming that the second derivative is bounded, and also give a priori error bounds. Moreover, we apply our results to the numerical solution of a non-linear boundary value problem of second-order.

Keywords: Convergence; A priori error bounds; Recurrence relations; Boundary value problems

1.Introduction

Most of the iterative methods of R-order four convergence need to calculate the second derivative, in this paper we will discuss an iterative method proposed by Ostrawski in [1] and Traub in [2]:

$$y_n = x_n - f'(x_n)^{-1} f(x_n)$$
$$x_{n+1} = y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n)$$

which has R-order four convergence but only need to calculate the first derivative.

Notice the divided difference of first-order for the operator f

$$f[x,y]=\frac{f(x)-f(y)}{x-y}$$

we get

$$y_n = x_n - f^{'}(x_n)^{-1}f(x_n)$$
$$x_{n+1} = y_n - \frac{f(y_n)}{2f[y_n,x_n]-f^{'}(x_n)}$$

In this paper we will discuss this iterative method in Banach spaces rather than 1st-dimensional real space in which we always discuss the divided difference.

Let X,Y be Banach spaces, and let $F:\Omega\subseteq \mathrm{X}\to \mathrm{Y}$ be a nonlinear operator and consider the equation

$$F(x)=0 \tag{1}$$

Let us denote by $\mathcal{L}(\mathrm{X},\mathrm{Y})$ the space of bounded linear operators from X to Y. An operator $[x,y;F]\in\mathcal{L}(\mathrm{X},\mathrm{Y})$ is called a divided difference of first-order for the operator F on the points x and $y(x\neq y)$ if the following equality holds:

$$[x,y;F](x-y)=F(x)-F(y).$$

Using this definition, the iteration above can be described by the following:

$$y_n = x_n - F^{'}(x_n)^{-1}F(x_n)$$
$$x_{n+1} = y_n - \Big([y_n,x_n;F]+[x_n,y_n;F]-F^{'}(x_n)\Big)^{-1}F(y_n) \tag{2}$$

Compared with the Newton's method it takes one time computation of derivatives but has almost the same convergence speed as Newton method.

The convergence of (2) to a solution of (1) has been usually studied from majorizing sequences [3,4]. In this paper, we analysis the convergence of (2) by using a technique that consists of a new system of recurrence relations [5,6]. And the use of these recurrence relations allows us to obtain a priori error bounds.

Further more we will apply this method to the numerical solution of a non-linear boundary value problem of second-order, it is faster than the method discussed in [5], and has the

same speed as the method in [7] but without calculating the second derivative.

2.Recurrence relations

We establish the recurrence relations from which the convergence of (2) is proved later.

Let X,Y be Banach spaces, and let $F : \Omega \subseteq \mathrm{X} \to \mathrm{Y}$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. Now we will study the convergence of (2) to a solution x^* of equation $F(x) = 0$.

In Section 3 and 4 we assume that

(i) $\|F^{'}(x_0)^{-1}\| \leq \beta$

(ii) $\|y_0 - x_0\| = \|F^{'}(x_0)^{-1}F(x_0)\| \leq \eta$

(iii) $\|F^{''}(x)\| \leq k$

Let us suppose that

$$a_0 = k\beta\eta$$

and let us define the sequence

$$a_n = f(a_{n-1})^2 g(a_{n-1}) a_{n-1}$$

where

$$f(x) = \frac{2(1-x)}{2-4x+x^2}, \qquad g(x) = \frac{x^3}{8(1-x)^2}$$

Notice that

$$k\|F^{'}(x_0)^{-1}\|\|F^{'}(x_0)^{-1}F(x_0)\| \leq a_0,$$

$$\|x_1 - x_0\| \leq \Big(1 + \frac{a_0}{2(1-a_0)}\Big)\|y_0 - x_0\|.$$

Given this situation, we prove, for $n \geq 1$, the following statement:

$(\mathrm{I_n})$ $\|F^{'}(x_n)^{-1}\| \leq f(a_{n-1})\|F^{'}(x_{n-1})^{-1}\|$

$(\mathrm{II_n})$ $\|y_n - x_n\| \leq \|F^{'}(x_n)^{-1}F(x_n)\| \leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\|$

$(\mathrm{III_n})$ $k\|F^{'}(x_n)^{-1}\|\|F^{'}(x_n)^{-1}F(x_n)\| \leq a_n$

$(\mathrm{IV_n})$ $\|x_{n+1} - x_n\| \leq \Big(1 + \frac{a_n}{2(1-a_n)}\Big)\|y_n - x_n\|$

Assuming

$$a_0\Big(1+\frac{a_0}{2(1-a_0)}\Big)<1 \qquad\qquad x_1\in\Omega_0,$$

then we have

$$\begin{aligned}\|I-F^{'}(x_0)^{-1}F^{'}(x_1)\| &\le \|F^{'}(x_0)^{-1}\|\|F^{'}(x_0)-F^{'}(x_1)\|\\ &\le k\|F^{'}(x_0)^{-1}\|\Big(\|x_1-y_0\|+\|y_0-x_0\|\Big)\\ &\le a_0\Big(1+\tfrac{a_0}{2(1-a_0)}\Big)<1\end{aligned}$$

so $F^{'}(x_1)^{-1}$ is defined and since

$$\begin{aligned}F(x_1)&=\int_0^1 F^{''}(\,y_0+\theta(x_1-y_0))(1-\theta)\mathrm{d}\theta(x_1-y_0)^2\\ &\qquad -[y_0,x_0;F]-[x_0,y_0;F](x_1-y_0)\\ &\qquad +F^{'}(x_0)(x_1-y_0)+F^{'}(y_0)(x_1-y_0)\\ &=\int_0^1 F^{''}(\,y_0+\theta(x_1-y_0))(1-\theta)\mathrm{d}\theta(x_1-y_0)^2\\ &\qquad +\int_0^1 F^{'}(y_0)-F^{'}(x_0+\theta(y_0-x_0))\mathrm{d}\theta(x_1-y_0)\\ &\qquad +\int_0^1 F^{'}(x_0)-F^{'}(x_0+\theta(y_0-x_0))\mathrm{d}\theta(x_1-y_0)\\ &=\int_0^1 F^{''}(\,y_0+\theta(x_1-y_0))(1-\theta)\mathrm{d}\theta(x_1-y_0)^2\end{aligned}$$

so

$$\begin{aligned}\|F^{'}(x_1)^{-1}F^{'}(x_1)\| &\le \|F^{'}(x_1)^{-1}F^{'}(x_0)\|\|F^{'}(x_0)^{-1}\|\|F(x_1)\|\\ &\le \frac{k}{2(1-a_0(1+\frac{a_0}{2(1-a_0)}))}\|F^{'}(x_0)^{-1}\|\|x_1-y_0\|^2\\ &\le \frac{1}{1-(1+\frac{a_0}{2(1-a_0)})a_0}\frac{a_0^3}{8(1-a_0)^2}\|y_0-x_0\|\\ &= f(a_0)g(a_0)\|y_0-x_0\|\end{aligned}$$

and

$$\begin{aligned}&\Big\|I-F^{'}(x_1)^{-1}\Big([y_1,x_1;F]+[x_1,y_1;F]-F^{'}(x_1)\Big)\Big\|\\ &\qquad\le \|F^{'}(x_1)^{-1}\|\|F^{'}(x_1)-[y_1,x_1;F]-[x_1,y_1;F]+F^{'}(x_1)\|\\ &\qquad\le k\|F^{'}(x_1)^{-1}\|\|y_1-x_1\|\le a_1<1\end{aligned}$$

so $\Big([y_1, x_1; F] + [x_1, y_1; F] - F^{'}(x_1)\Big)^{-1}$ is defined and since

$$F(y_1) = \int_0^1 F^{''}(x_1 + \theta(y_1 - x_1))(1-\theta)\mathrm{d}\theta(y_1 - x_1)^2$$

we get

$$\begin{aligned}\|x_2 - y_1\| &= \Big\|\Big([y_1, x_1; F] + [x_1, y_1; F] - F^{'}(x_1)\Big)^{-1} F(y_1)\Big\| \\ &\leq k\frac{1}{1-a_1}\|F^{'}(x_1)^{-1}\|\frac{1}{2}\|y_1 - x_1\|^2 \\ &= \frac{a_1}{2(1-a_1)}\|y_1 - x_1\|\end{aligned}$$

Now following an inductive procedure and assuming that

$$x_n, y_n \in \Omega_0, \quad a_n\Big(1 + \frac{a_n}{2(1-a_n)}\Big) < 1, \quad n \in \mathbf{N} \tag{3}$$

items $(\mathrm{I}_n) - (\mathrm{IV}_n)$ are proved.

Now we must analyze the real sequence $\{a_n\}$ to study the sequence $\{x_n\}$ defined in a Banach space. To establish the convergence of $\{x_n\}$, we have only to prove that it is a Cauchy sequence and prove the above assumption (3).

3.Convergence Study

In this section, we study sequence $\{a_n\}$ defined above, to prove the convergence of the sequence $\{x_n\}$ given by (1). First of all, we give a technical lemma including results concerning functions of one variables, which will be needed later.

Lemma 3.1 *Under the previous notations, we have that $f(x)$ is increasing and $f(x) > 1$ for $x \in (0, 0.5)$, and the same to $g(x)$. Moreover, if $\gamma \in (0,1)$, then $g(\gamma x) \leq \gamma^3(x)$, for $x \in (0, 0.5)$*

Lemma 3.2 *Let $0 < a_0 < 0.5$ and $f(a_0)^2 g(a_0) < 1$. Then, the sequence $\{a_n\}$ is decreasing.*

Proof. From the hypotheses, we deduce that $0 < a_1 < a_0$, since $f(x) > 1$ in $(0, 0.5)$. Now, we suppose that $0 < a_k < a_{k-1} < \cdots < a_1 < a_0 < 0.5$. Then, $0 < a_{k+1} < a_k$ if and only if

$f(a_k)^2 g(a_k) < 1$. Notice that

$$f(a_k) < f(a_0), \qquad g(a_k) < g(a_0).$$

Consequently,

$$f(a_k)^2 g(a_k) < 1.$$

Then the result holds.

In the following lemma, whose proof is obvious,we give sufficient conditions so that the real sequence $\{a_n\}$ is decreasing.

Lemma 3.3 *If* $0 < a_0 < 0.5$ *then* $f(a_0)^2 g(a_0) < 1$.

The next step is to prove (3). Under the hypotheses of the previous lemma, we have that,

$$a_n\Big(1 + \frac{a_n}{2(1-a_n)}\Big) < a_0\Big(1 + \frac{a_0}{2(1-a_0)}\Big) < 1, \quad \text{if and only if } \frac{a_0^2 - 4a_0 + 2}{2(a_0 - 1)} < 0.$$

This inequality is true since $a_0 \in (0, 0.5)$.

We will now prove that $(1 + \frac{a_n}{2(1-a_n)})\|y_n - x_n\|$ is a Cauchy sequence. We note that

$$\begin{aligned}
&\Big(1+ \frac{a_n}{2(1-a_n)}\Big)\|y_n - x_n\| \\
&\quad \le \Big(1 + \frac{a_0}{2(1-a_0)}\Big) f(a_{n-1}) g(a_{n-1}) \|y_{n-1} - x_{n-1}\| \\
&\quad \le \cdots \le \Big(1 + \frac{a_0}{2(1-a_0)}\Big) \|F^{'}(x_0) F(x_0)\| \prod_{k=0}^{n-1} f(a_k) g(a_k).
\end{aligned}$$

We analyze next the factor

$$\prod_{k=0}^{n-1} f(a_k) g(a_k)$$

by means of the following lemma.

Lemma 3.4 *Let us define* $\gamma = a_1/a_0$. *Then,*

(i) $\gamma = f(a_0)^2 g(a_0) \in (0, 1)$,

(ii$_n$) $a_n \le \gamma^{4^{n-1}} a_{n-1} \le \gamma^{(4^n - 1)/3} a_0$,

(iii$_n$) $f(a_n) g(a_n) \le \gamma^{4^n} [f(a_0) g(a_0)/\gamma] = \gamma^{4^n}/f(a_0)$.

Proof. Notice that (i) is trivial. We prove (ii_n) by following an inductive procedure. We have,

$$a_1 \leq \gamma a_0$$

and by Lemma 3.1 the result holds. If we suppose that (ii_n) is true, then

$$\begin{aligned} a_{n+1} &= a_n f(a_n)^2 g(a_n) \\ &\leq \gamma^{4^{n-1}} a_{n-1} f(\gamma^{4^{n-1}} a_{n-1})^2 g(\gamma^{4^{n-1}} a_{n-1}) \\ &\leq \gamma^{4^{n-1}} a_{n-1} f(\gamma^{4^{n-1}} a_{n-1})^2 (\gamma^{4^{n-1}})^3 g(a_{n-1}) = \gamma^{4^n} a_n. \end{aligned}$$

Now as $a_{n+1}/a_n \leq \gamma^{4^n}$, ($\text{ii}_\text{n}$) also holds. Moreover,

$$a_{n+1} \leq \gamma^{4^n} a_n \leq \gamma^{4^n} \gamma^{4^{n-1}} a_{n-1} \leq \cdots \leq \gamma^{(4^{n+1}-1)/3} a_0.$$

Finally, we observe that

$$\begin{aligned} f(a_n)g(a_n) &\leq f(\gamma^{(4^n-1)/3} a_0) g(\gamma^{(4^n-1)/3} a_0) \\ &\leq \gamma^{4^n} [f(a_0 g(a_0)/\gamma] \\ &= \gamma^{4^n}/f(a_0), \end{aligned}$$

and the proof is complete. □

As a consequence of all the above, if we denote $\Delta = 1/f(a_0)$, it follows that

$$\prod_{k=0}^{n-1} f(a_k)g(a_k) \leq \prod_{k=0}^{n-1} (\gamma^{4^n} \Delta) = \gamma^{(4^n-1)/3} \Delta^n.$$

So, from $\Delta < 1$, we deduce that $\prod_{k=0}^{n-1} f(a_k)g(a_k)$ converges to zero by letting $n \to \infty$.

We are now ready to state the following result on convergence for (1).

Theorem 3.1 *Let* X,Y *be Banach spaces, and let* $F : \Omega \subseteq \text{X} \to \text{Y}$ *be a nonlinear twice Fréchet differentiable operator in an open convex domain* $\Omega_0 \subseteq \Omega$. *Let us assume that (i)-(iv) are satisfied. Let us denote* $a_n = k\beta\eta$. *Suppose that* $0 < a_0 < 0.5$. *Then if* $\overline{B(x_0, R\eta)} = \{x \in \text{X}; \|x - x_0\| \leq R\eta\} \subseteq \Omega_0$, *where* $R = (1 + \frac{a_0}{2(1-a_0)})\frac{1}{1-\Delta}$ *and* $\Delta = 1/f(a_0)$, *the sequence*

$\{x_n\}$ defined (1) and starting at x_0 converges R-cubically to a solution x^* of the equation $F(x) = 0$. In this case, the solutio x^* and the iterates y_n, x_n belong to $\overline{B(x_0, R\eta)}$. Moreover, the solution x^* is unique in $B(x_0, 2/k\beta - R\eta) \cap \Omega_0$. Furthermore, we can give the following error estimates:

$$\|x^* - x_n\| \leq \Big(1 + \frac{a_0}{2(1-a_0)}\gamma^{(4^n-1)/3}\Big)\gamma^{(4^n-1)/3}\Big[\frac{\Delta^n}{1-\Delta}\Big]\eta \tag{4}$$

Proof. When $0 < a_0 < 0.5$, the convergence of the sequence $\{x_n\}$ follows immediately from the previous lemmas. We consider $p \geq 1$ and

$$\begin{aligned}
&\| x_{n+p} - x_n\| \\
&\leq \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq \Big(1 + \frac{a_{n+p-1}}{2(1-a_{n+p-1})}\Big)\eta \prod_{j=0}^{n+p-2} f(a_j)g(a_j) + \cdots + \Big(1 + \frac{a_n}{2(1-a_n)}\Big)\eta \prod_{j=0}^{n-1} f(a_j)g(a_j) \\
&\leq \Big(1 + \frac{a_n}{2(1-a_n)}\Big)\Big[\gamma^{(4^{n+p-1}-1)/3}\Delta^{n+p-1} + \cdots + \gamma^{(4^n-1)/3}\Delta^n\Big]\eta \\
&\leq \Big(1 + \frac{a_0}{2(1-a_0)}\gamma^{(4^n-1)/3}\Big)\gamma^{(4^n-1)/3}\Big[\frac{\Delta^n(1-\Delta^p)}{(1-\Delta)}\Big]\eta
\end{aligned} \tag{5}$$

Therefore, we obtain

$$\|x_p - x_0\| \leq \Big(1 + \frac{a_0}{2(1-a_0)}\Big)\Big[\frac{1-\Delta^p}{1-\Delta}\Big]\eta < R\eta,$$

for $n = 0$. By letting $p \to \infty$ in (5), we also get (4). Similarly, we infer that y_n belongs to $B(x_0, R\eta)$.

To prove that $F(x^*) = 0$, notice that

$$\|F^{'}(x_n)^{-1}F(x_n)\| \to 0, \qquad as \;\; n \to \infty.$$

Since

$$\|F(x_n)\| \leq \|F^{'}(x_n)\| \cdot \|F^{'}(x_n)^{-1}F(x_n)\|$$

and $\{\|F^{'}(x_n)\|\}$ is a bounded sequence, we deduce that $\|F(x_n)\| \to 0$ and then $F(x^*) = 0$ from the continuity of F.

To show uniqueness, suppose that $y^* \in B(x_0, 2/k\beta - R\eta) \cap \Omega_0$ is another solution of $F(x) = 0$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F^{"}(x^* + \theta(y^* - x^*))\mathrm{d}\theta(y^* - x^*).$$

Using the estimate

$$\begin{aligned}
\|F^{'}(x_n)^{-1}\| \int_0^1 \|F^{'}(x^* + \theta(y^* - x^*)) - F^{'}(x_0)\|\mathrm{d}\theta \\
&\leq k\beta \int_0^1 \|x^* + \theta(y^* - x^*) - x_0\|\mathrm{d}\theta \\
&\leq k\beta \int_0^1 [(1-\theta)\|x^* - x_0\| + \theta\|y^* - x_0\|]\mathrm{d}\theta \\
&< (k\beta/2)(R\eta + 2/k\beta - R\eta) \\
&= 1
\end{aligned}$$

we infer that the operator $\int_0^1 F^{'}(x^* + t(y^* - x^*))\mathrm{d}t$ has an inverse, and consequently, $y^* = x^*$.

Finally, from (4), we deduce that the R-order of convergence of the sequence (1) is four,

$$\|x^* - x_n\| \leq \Big(1 + \frac{a_0}{2(1-a_0)}\Big)\gamma^{-1/3}\Big[\frac{\eta}{1-\Delta}\Big](\gamma^{1/3})^{4^n}, \quad \gamma < 1$$

The proof is complete. □

4.Numerical example

Now we apply the semilocal convergence result given above to an example also considered in [5]. We consider the following non-linear boundary value problem of second-order:

$$\begin{aligned}
&x^{''} + x^{1+p} = 0, \qquad p \in (0,1) \\
&x(0) = x(1) = 0.
\end{aligned} \tag{6}$$

To solve this problem by finite differences, we start by drawing the usual grid line with grid points $t_i = ih$, where $h = 1/n$ and n is an appropriate integer. Note that x_0 and x_n are given by the boundary conditions, then $x_n = 0 = x_n$. We first approximate the second derivative $x^{''}(t)$ in the differential equation by

$$x''(t) \approx [x(t+h) - 2x(t) + x(t-h)]/h^2,$$

$$x''(t_i) = (x_{i+1} - 2x_i + x_{i-1})/h^2, \quad i = 1, 2, \cdots, n-1.$$

By substituting this expression into the differential equation, we have the following system of non-linear equations:

$$\begin{aligned} &2x_1 - h^2 x_1^{1+p} - x_2 = 0 \\ &-x_{i-1} + 2x_i - h^2 x_i^{1+p} - x_{i+1} = 0, \quad i = 2, 3, \cdots, n-2, \\ &-x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^{1+p} = 0. \end{aligned} \tag{7}$$

We therefore have an operator $F : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ such that $F(x) = H \cdot x - h^2 g(x)$, where

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} x_1^{1+p} \\ x_2^{1+p} \\ \vdots \\ x_{n-1}^{1+p} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Thus

$$F'(x) = H - h^2(1+p)\mathrm{diag}\{x_1^p, x_2^p, \cdots, x_{n-1}^p\}.$$

Let $x \in \mathbf{R}^{n-1}$. Then our norm will be $||x|| = \max\limits_{1 \le i \le n-1} ||x_i||$. The corresponding norm on $A \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ is

$$||A|| = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} ||a_{ij}||.$$

It is known (see [8,9]) that F has a divided difference at the points $u, v \in \mathbf{R}^{n-1}$, which is defined by the matrix whose entries are

$$\begin{aligned} [u, v; F]_{ij} = \quad & \frac{1}{u_j - v_j}(F_i(u_1, \cdots, u_j, v_{j+1}, \cdots, v_{n-1}) \\ & -F_i(u_1, \cdots, u_{j-1}, v_j, \cdots, v_{n-1})). \end{aligned}$$

Therefore

$$[u,v;F] = H - h^2 \begin{pmatrix} \dfrac{u_1^{1+p} - v_1^{1+p}}{u_1 - v_1} & 0 & \cdots & 0 \\ 0 & \dfrac{u_2^{1+p} - v_2^{1+p}}{u_2 - v_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{u_{n-1}^{1+p} - v_{n-1}^{1+p}}{u_{n-1} - v_{n-1}} \end{pmatrix}.$$

In this case, we have that $[u,v;F] = \int_0^1 F^{'}(u+\theta(u-v))\mathrm{d}\theta$. So we study the value $||F^{'}(x) - F^{'}(u)||$ to obtain a bound for $|||[x,y;F] - [u,v;F]|||$.

For all with $x, u \in \mathbf{R}^{n-1}$,con$|x_i| > 0, |u_i| > 0, (i = 1,2,\cdots,n-1)$, and taking into account the max-norm it follows:

$$\begin{aligned} ||F^{'}(x) - F^{'}(u)|| &= ||\mathrm{diag}\{h^2(1+p)(u_i^p - x_i^p)\}|| \\ &= h^2(1+p)\max_{1\le i\le n-1}|u_i^p - x_i^p| \\ &\le h^2(1+p)[\max_{1\le i\le n-1}|u_i - x_i|]^p \\ &= h^2(1+p)||u-x||^p. \end{aligned}$$

Therefore

$$\begin{aligned} |||[x,y;F] - [u,v;F]||| &\le \int_0^1 ||F^{'}(x+\theta(y-x)) - F^{'}(u+\theta(v-u))||\mathrm{d}\theta \\ &\le h^2(1+p)\int_0^1 ||(1-\theta)(x-u) + \theta(y-v)||^p\mathrm{d}\theta \\ &\le h^2(1+p)\int_0^1 ((10\theta)^p||x-u||^p + \theta^p||y-v||^p)\mathrm{d}\theta \\ &= h^2(||x-u||^p + ||y-v||^p), \end{aligned}$$

so $k = 2h^2$.

Now we apply the R-order four method to approximate the solution of $F(x) = 0$.

If we choose $p = 1/2$ and if $n = 10$, then (7)gives nine equations. We choose x_0 as

$$x_0 = \begin{pmatrix} 33.57498274928053 \\ 65.20452867501265 \\ 91.56893412724006 \\ 109.1710943553677 \\ 115.3666988182897 \\ 109.1710943553677 \\ 91.56893412724006 \\ 65.20452867501265 \\ 33.57498274928053 \end{pmatrix}$$

With the notation of Theorem 3.1 we can easily obtain the following results:

$$\beta = 26.5876, \qquad \eta = 0.00366509, \qquad R\eta = 1.88058$$

$$a_0 = 0.00194892 < 0.5, \qquad f(a_0)^2 g(a_0) = 9.32570911847766481 \times 1^{-10} < 1$$

so the hypotheses of Lemma 3.2 are satisfied. We obtain by Theorem 3.1 that the sequence $\{x_n\}$ given by the R-order four method converges to a solution x^* in $\overline{B(x_0, R\eta)}$ of equation $F(x) = 0$. After one step we get the vector x^* as the solution of system (7):

$$x^* = \begin{pmatrix} 33.57391204833779 \\ 65.20245092365435 \\ 91.56602003553957 \\ 109.1676242966423 \\ 115.3630336377466 \\ 109.1676242966423 \\ 91.56602003553957 \\ 65.20245092365435 \\ 33.57391204833779 \end{pmatrix}$$

and if we use the Secant method in [5], then we need 3 steps.

And if we choose $p = 1$ and $n = 10$ and

$$x_0 = \begin{pmatrix} 0.0772542 \\ 0.1469460 \\ 0.2022540 \\ 0.2377640 \\ 0.2500000 \\ 0.2377640 \\ 0.2022540 \\ 0.1469460 \\ 0.0772542 \end{pmatrix}$$

([7]) then we get the solution

$$x^* = \begin{pmatrix} -8.026098 \times 10^{-7} \\ -1.578568 \times 10^{-6} \\ -2.251941 \times 10^{-6} \\ -2.71847 \times 10^{-6} \\ -2.886245 \times 10^{-6} \\ -2.71847 \times 10^{-6} \\ -2.251941 \times 10^{-6} \\ -1.578568 \times 10^{-6} \\ -8.026098 \times 10^{-7} \end{pmatrix}$$

without calculating the second derivative.

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Numerical Treatment Of The Two-dimensional Heat Radiation Integral Equation

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The radiation exchange in both convex and non-convex enclosures of diffuse gray surfaces is given in the form of a Fredholm boundary integral equation of the second kind. A boundary element method which is based on the Galerkin discretization schem is implemented for this integral equation. Four iterative methods are used to solve the linear system of equations resulted from the Galerkin discretization scheme. A comparison is drawn between these methods.
Theoretical error estimates for the Galerkin method has shown to be in a good agreement with numerical experiments.

KEYWORDS: Fredholm integral equation; heat radiation; iterative methods; error estimations.

1. INTRODUCTION

Transport of heat radiative energy between two points in convex or non-convex enclosures of diffuse gray surfaces is one of the few phenomena that are often governed directly by an integral equation. One of the consequences of this fact is that the pencil of rays emitted at one point can impinge another point only if these two points can "see" each other, i.e. the line segment connected these points does not intersect any surface. The presence of the shadow zones should be taken into account in heat radiation analysis whenever the domain where the radiation heat transfer is taking place, is not convex.

Shadow zones computation in some respect is not easy, but we were able to develop an efficient algorithm for this purpose and was implemented in our computer program. The integral equation governing the heat radiation (see section 2 for the formulation of the problem) was earlier solved for the convex enclosure using the Monte Carlo method [5].

In [2, 3] a boundary element method was implemented to obtain a direct numerical solution for this integral equation. This latter method permits quite high error bounds. For two-dimensional enclosure and three-dimensional rotational symmetric convex enclosure a Panel method has been developed [9] and then coupled with heat transport through radiation and conduction.

In this paper we are concerned to use the boundary element method, which is regarded to be the most popular numerical method for solving this type of problems. Thus we will present an efficient and reliable iterative methods to solve the linear system arises from Galerkin discretization scheme for the boundary integral equation. Numerical results for both convex and non-convex geometries have been obtained. We will present some error estimates for the Galerkin discretization method. Theoretically Galerkin method requires a time consuming double integral over Γ for the calculation of every element of the stiffness matrix. Thus we choose the corresponding numerical Gaussian quadrature formula with respect to a fast computation, i.e. by evaluating the kernel of the integral equation as seldom as possible. Numerical experiments with examples show high accuracy and efficiency of this method. The theoretical asymptotic error estimates are in rather good agreement with numerical experiments.

2. THE FORMULATION OF THE TWO DIMENSIONAL HEAT RADIATION PROBLEM

We consider an enclosure $\Omega \subset \mathbb{R}^2$ with boundary Γ. The boundary of the enclosure is composed of N elements as shown in Fig. 1.

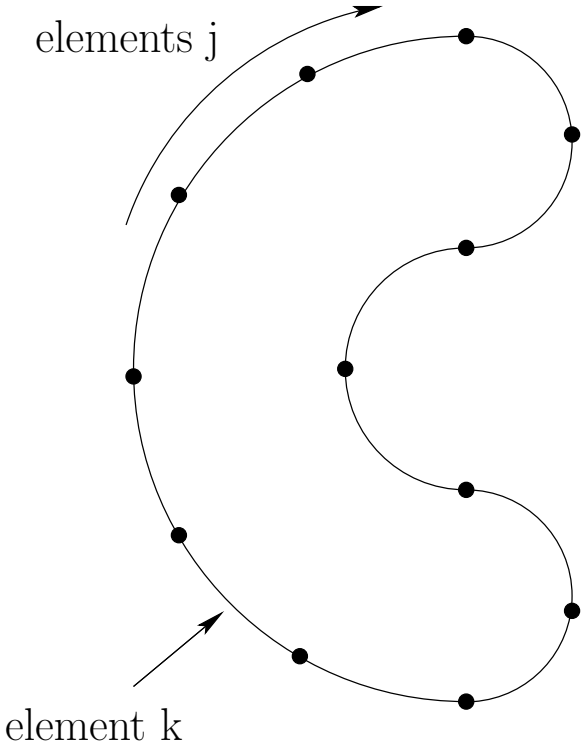


Fig.1

The heat balance for an element k with area dA_k reads as

$$Q_k = q_k dA_k = (q_{0,k} - q_{i,k}) dA_k, \tag{2.1}$$

where

$q_{i,k}$: is the rate of incomming radiant energy per unit area on the element k.
$q_{o,k}$: is the rate of outgoing radiant energy per unit area on the elment k.
dA_k : is the area of element k.
q_k : is the energy flux supplied to the element k by some means other than the radiation inside the enclosure to make up for the net radiation loss and maintain the specified inside surface temperature.

A second equation results from the fact that the energy flux leaving the surface is composed of emitted plus reflected energy. This yields to

$$q_{0,k} = \varepsilon_k \sigma T_4^k + l_k q_{i,k} \tag{2.2}$$

where

ε_k : is the emissivity coefficient $(0 < \varepsilon_k < 1)$.
σ : is the Stefan-Boltzmann constant which has the value $5.6696 \times 10^{-8}\ W/m^2K$.
l_k : is the reflection coefficient with the reation $l_k = 1 - \varepsilon_k$ is used for a gray surfaces.

The incident flux $q_{i,k}$ is composed of the portion of the energy leaving the viewable surfaces of the enclosure and arriving at the k-th surface. If the k-th surface can view itself (is non convex), a portion of its outgoing flux will contribute directly to its incidient flux. The incidient energy is then equal

$$\begin{aligned} dA_k q_{i,k} &= dA_1 q_{0,1} F_{1,k}\beta(1,k) + dA_2 q_{0,2} F_{2,k}\beta(2,k) + \ldots \\ &\quad + dA_j q_{0,j} F_{j,k}\beta(j,k) + \ldots + dA_k q_{0,k} F_{k,k}\beta(k,k) + \ldots \\ &\quad + dA_N q_{0,N} F_{N,k}\beta(N,k). \end{aligned} \tag{2.3}$$

From the view factor reciprocity relation [11] follows

$$\left.\begin{aligned} dA_1 F_{1,k}\beta(1,k) &= dA_k F_{k,1}\beta(k,1) \\ dA_2 F_{2,k}\beta(2,k) &= dA_k F_{k,2}\beta(k,2) \\ &\vdots \\ dA_N F_{N,k}\beta(N,k) &= dA_k F_{k,N}\beta(k,N) \end{aligned}\right\} \tag{2.4}$$

Then equation (2.3) can be rewritten in such a way that the only area appearing is dA_k:

$$\begin{aligned} dA_k q_{i,k} &= dA_k F_{k,1}\beta(k,1) q_{0,1} + dA_k F_{k,2}\beta(k,2) q_{0,2} + \ldots \\ &\quad + dA_k F_{k,j}\beta(k,j) q_{0,j} + \ldots + dA_k F_{k,k}\beta(k,k) q_{0,k} + \ldots \\ &\quad + dA_k F_{k,N}\beta(k,N) q_{0,N}. \end{aligned} \tag{2.5}$$

so that the incident flux can be expressed as

$$q_{i,k} = \sum_{j=1}^{N} F_{k,j}\beta(k,j) q_{0,j} \tag{2.6}$$

The visibility factor $\beta(k,j)$ is defined as (see for example [12])

$$\beta(k,j) = \begin{cases} 1 & \text{when there is a heat exchange between the} \\ & \text{surface element } k \text{ and the surface element } j \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

Substituting (2.6) into (2.2) and using the relation $l_k = 1 - \varepsilon_k$ yields

$$q_{0,k} = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k) \sum_{j=1}^{N} F_{k,j} \beta(k,j) q_{0,j}. \tag{2.8}$$

2.1. The Calculation Of The View Factor $F_{k,j}$

The total energy per unit time leaving the surface element dA_k and incident at the element dA_j is given through

$$Q_{k,j} = L_k dA_k \cos(\theta_k) d\omega_k, \tag{2.9}$$

where $d\omega_k$ is the solid angle subtended by dA_j when viewed from dA_k (see Fig.2) and L_k is the total intensity of a black body leaving the element dA_k.

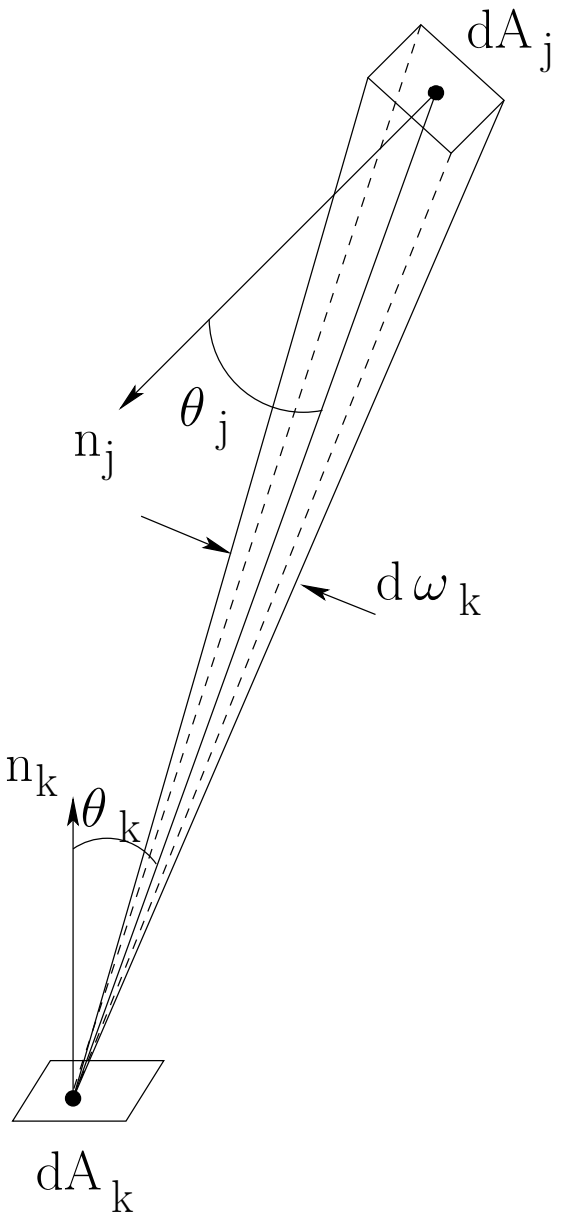


Fig.2

The solid angle $d\omega_k$ is related to the projected area dA_k and the distance $S_{k,j}$ between the elements dA_k and dA_j and can be calculated as

$$d\omega_k = \frac{dA_j \cos(\theta_j)}{S_{k,j}^2}, \tag{2.10}$$

where θ_j denotes the angle between the normal vector n_j and the distance vector $S_{k,j}$. Substituting (2.10) into (2.9) gives the following equation for the total energy per unit time leaving dA_k and arriving at dA_j:

$$Q_{k,j} = \frac{L_k dA_k \cos(\theta_k) dA_j \cos(\theta_j)}{S_{k,j}^2} \tag{2.11}$$

In [12], we have the relation between the total intensity E_k of a black body i.e.,

$$L_k = \frac{E_k}{\pi} = \frac{\sigma T_k^4}{\pi} \tag{2.12}$$

and consequantly equation (2.11) becomes

$$Q_{k,j} = \frac{\sigma T_k^4 \cos(\theta_k) \cos(\theta_j) dA_k dA_j}{\pi S_{k,j}^2}. \tag{2.13}$$

From the definition of the view factor $F_{k,j}$ (see [11]) together with (2.13), we get

$$F_{k,j} = \frac{Q_{k,j}}{\sigma T_k^4 dA_k} = \frac{\cos(\theta_k) \cos(\theta_j) dA_j}{\pi S_{k,j}^2}. \tag{2.14}$$

2.2. The Boundary Integral Equation

Now we are able to derive the boundary integral equation describing the heat balance in a gray body. The substitution of equation (2.14) into equation (2.8) leads to

$$q_{0,k} = \varepsilon_k \sigma T_k^4 + (1 - \varepsilon_k) \sum_{j=1}^{N} \frac{\cos(\theta_k) \cos(\theta_j) dA_j}{\pi S_{k,j}^2} \beta(k,j) q_{0,j}. \tag{2.15}$$

If the number of the area elements $N \to \infty$, then for all $x \in dA_k$ we obtain the following boundary integral equation

$$q_0(x) = \varepsilon(x) \sigma T^4(x) + (1 - \varepsilon(x)) \int_\Gamma G(x,y) q_0(y) d\Gamma_y \text{ for } x \in \Gamma, \tag{2.16}$$

where the kernel $G(x,y)$ denotes the view factor between the points x and y of Γ. From the above consideration and for general enclosure geometries $G(x,y)$ is given through

$$G(x,y) := G^*(x,y)\beta(x,y) = \frac{[n(y)\cdot(y-x)]\cdot[n(x)\cdot(x-y)]\cdot\beta(x,y)}{2|x-y|^3}. \qquad (2.17)$$

For convex enclosure geometries $\beta(x,y) \equiv 1$. If the enclosure ist not convex then we have to take into account the visibility function $\beta(x,y)$:

$$\beta(x,y) = \begin{cases} 1 & \text{for} \quad n(y)\cdot(y-x) \wedge n(x)\cdot(x-y) > 0 \wedge \vec{xy} \cap \Gamma = \emptyset \\ 0 & \text{for} \quad \vec{xy} \cap \Gamma \neq \emptyset \end{cases} \qquad (2.18)$$

where $\vec{xy}$ denotes the open straight segment between the points x and y. Definition (2.18) implies that $\beta(x,y) = \beta(y,x)$. Since $G^*(x,y)$ is symmetric then $G(x,y)$ is also symmetric.
Equation (2.16) is a Fredholm boundary integral equation of the second kind. We introduce the integral operator $\widetilde{K} : L^\infty(\Gamma) \to L^\infty(\Gamma)$ with

$$\widetilde{K}q_0(x) := \int_\Gamma G(x,y)q_0(y)d\Gamma_y \qquad \text{for } x \in \Gamma, q_0 \in L^\infty(\Gamma). \qquad (2.19)$$

Some of the properties of the integral operator (2.19) along with the solvability of equation (2.16) have been investigated in [12].

3. NUMERICAL APPROXIMATION TO THE SOLUTION OF THE FREDHOLM INTEGRAL EQUATION

3.1. Boundary Element Method and Galerkin Discretization

In a two-dimensional case we let Γ be a curve that is given by a regular parameter representation [10]

$$\Gamma : y = Z_j(t) \quad \text{for } t \in \mathbb{R},\ j = 1,\ldots,L \qquad (3.1)$$

We choose on $\mathbb{R}$ a family of 1-periodic interval partition:

$$0 = t_0 < t_< \cdots < t_N = 1,$$
$$\Pi_h = \{t_k\}_{-\infty}^{\infty},\ t_{k+N} = t_k + 1 \text{ with } h = max\{t_{k+1} - t_k\} \to 0. \qquad (3.2)$$

Let $S_h^{d,r}$ be a family of 1-periodic piecewise polynomials of degree $(d-1)$ with respect to the partition Π_h in the sense of Babuska an Aziz [1] which is $(r-1)$ times continuous and differntiable. We denote with $\Phi_k(t)$ the basis trial functions with a smallest possible support (B-splines) (see Fig.3).

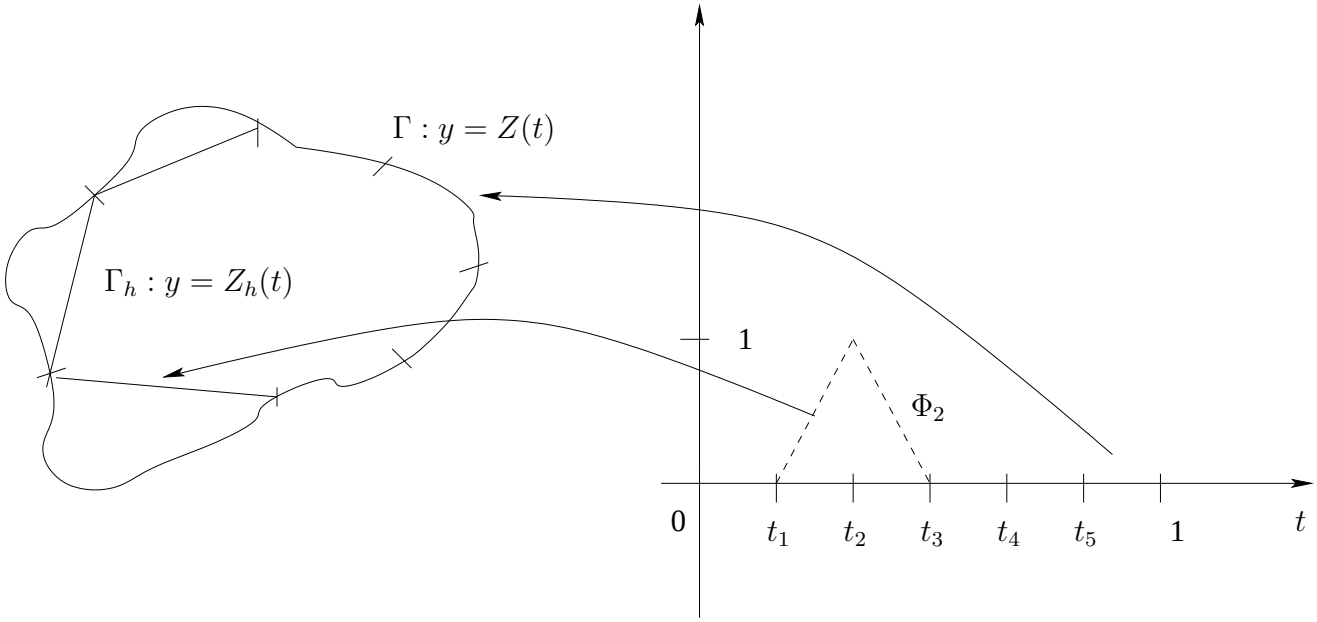


Fig. 3

The approximate solution has the general form

$$q_h(t) = \sum_{k=1}^{n} q_k \Phi_{k,n}(t) \qquad (3.3)$$

where n is the number of free grids and $q_k \in \mathbb{R}$, $k = 1, \ldots, n$ are the partition coefficients.

On partition in the parameter domain we use $S_h^{m+1,1}$-Lagrange-System of finite elements. Then the local representation of Γ transplant these finite element functions onto Γ_h. The ansatz functions (3.3) on Γ_h will then be defined by

$$\Gamma_h : y = Z_{jh}(t) \qquad (3.4)$$

with $Z_{jh}(t) = Z_j(t_k)$.
The ansatz functions (3.3) have the following approximation property

Approximation Property:
Let $\sigma \leq \tau \leq d$ be fulfilled and, for

$$\sigma < r + \frac{1}{2},\ \sigma < \frac{3}{2} \tag{3.5}$$

with the boundary approximation Γ_h, then there exists a constant c depending only on τ, σ and r and to any $v \in H^\tau(\Gamma)$ and any $S_h^{d,r}$ of our family there exists a finite element $\chi_h \in S_h^{d,r}$ such that

$$\|v - \chi_h\|_{H^\sigma(\Gamma)} \leq ch^{\tau-\sigma}\|v\|_{H^\tau(\Gamma)}. \tag{3.6}$$

Sometimes we shall additionally use the inverse property which holds for regular families $S_h^{d,r}$ subject to quasi-uniform of meshes.

Inverse Property:
For $\tau \leq \sigma$ with (3.5) there holds an estimate

$$\|\chi_h\|_{H^\sigma(\Gamma)} \leq c^* h^{\tau-\sigma}\|\chi_h\|_{H^\tau(\Gamma)} \quad \text{for } \chi_h \in S_h^{d,r} \tag{3.7}$$

where the constant c^* is independent of χ_h and h.

3.1.1. Representation Of System Of Equations

The Fredholm integral equation (2.16) can be expressed as

$$q = g + Kq \tag{3.8}$$

where $Kq = (1-\varepsilon)\widetilde{K}q$ and

$$\widetilde{K}q(x) = \int_\Gamma G(x,y)q(y)d\Gamma_y \quad \text{for } x \in \Gamma \text{ and } q \in L^\infty(\Gamma) \tag{3.9}$$

We let

$$\langle u, v\rangle_\Gamma := \int_0^1 u(t)v(t)|\dot{x}(t)|dt.$$

The Galerkin discretization of the integral equation (2.16) with the ansatz function (3.3) is given by

$$\sum_{k=1}^{n} q_k \langle \Phi_{k,n}, \Phi_{l,n} \rangle_\Gamma = \langle g, \Phi_{l,n} \rangle_\Gamma + \sum_{k=1}^{n} q_k \langle K\Phi_{k,n}, \Phi_{l,n} \rangle_\Gamma \tag{3.10}$$

Equation (3.10) can be written in the following short form:

$$(A_n - B_n)a_n = b_n \tag{3.11}$$

using the abbreviation $A = (q_{l,k})_{l,k=1,\ldots,n}$ for the mass matrix, with

$$q_{l,k} = \langle \Phi_{k,n}, \Phi_{l,n} \rangle_\Gamma = \int_0^1 \Phi_{l,n}(t)\Phi_{k,n}(t)|\dot{x}(t)|dt, \tag{3.12}$$

$B = (B_{l,k})_{l,k=1,\ldots,n}$ for the view factor matrix with

$$B_{l,k} = \langle K\Phi_{k,n}, \Phi_{l,n} \rangle_\Gamma = \int_0^1 \int_0^1 (1 - \varepsilon(t))\Phi_{l,n}(t)G(t,\tau)\Phi_{k,n}(\tau)|\dot{x}(t)||\dot{x}(\tau)|dtd\tau \tag{3.13}$$

and the vectors $a = (q_k)_{k=1,\ldots,n}$ and $b = \langle g, \Phi_{l,n} \rangle_\Gamma$, $l = 1, \ldots, n$.

Properties Of The Matrices

The mass matrix A in (3.11) is symmetric, positive definite and diagonal dominant hence it is invertible. Let λ_{min} and λ_{max} be the minimum and the maximum eigenvalues of the matrix A respectively then follows the well known estimations

$$\lambda_{min}\|q\|_{l^2}^2 \leq (A_n q, q) \leq \lambda_{max}\|q\|_{l^2}^2 \tag{3.14}$$

$$\frac{1}{\lambda_{max}}\|q\|_{l^2}^2 \leq (A_n^{-1} q, q) \leq \frac{1}{\lambda_{min}}\|q\|_{l^2}^2 \tag{3.15}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product of $\mathbb{R}^n$ with $(q,q) = \|q\|_{l^2}^2$.

Furthermore holds

$$\|A_n\|_{l^2} = \lambda_{max}, \quad \frac{1}{\|A_n^{-1}\|_{l^2}} = \lambda_{min}. \tag{3.16}$$

Also the system of equations $(A_n - B_n)$ is symmetric and positve definite.
Since the mass matrix A is invertible, equation (3.11) can then be expressed in the form

$$(I - A_n^{-1}B_n)a_n = A_n^{-1}b_n \tag{3.17}$$

Equation (3.17) can also be written as

$$q_n = g_n + K_n q_n, \tag{3.18}$$

where $q_n = a_n, g_n = A_n^{-1}b_n$ and $K_n = A_n^{-1}B_n$.

3.1.2. Hiearchie Discretized Problem

The discretization parameter n defines in general the dimension of the problem. For the multi-grid method we use the hiearchie of discretization in multi levels. For each stepwise h_l there is a corresponding parameter n_l. Hence the discretized vector equation of level l has the form

$$q_{n_l} = g_{n_l} + K_{n_l} q_{n_l} \tag{3.19}$$

To avoid the double indices n_l, we use for short

$$q_l = g_l + K_l q_l \qquad (l \geq 0) \tag{3.20}$$

where $q_l = a_l, g_l = A_l^{-1}b_l$ and $K_l = A_l^{-1}B_l$.

3.2. Solution Methods

To solve equation (3.20) we use four approximate iterative methods. These are the Picard-iteration or Neumann series method, two-grid and multi-grid methods and the conjugate gradient method.

3.2.1. Picard-Iteration

This is one of the iterative approximate method in which the pre-iteration step

$$q_l^{i+1} = g_l + K_l q_l^i \tag{3.21}$$

with the iteration step index i is directly obtained from the linear system of equations. It converges if and only if the spectral radius

$$\rho(K_l) < 1 \tag{3.22}$$

holds [7].
A sufficient condition for the convergence of this iteration method is

$$\|K_l\| < 1 \tag{3.23}$$

for a suitable matrix norm.

3.2.2. Two-Grid Method

The usual two-grid iteration of level l for one iteration step $q_l^i \to q_l^{i+1}$:

Smoothing step: $$q_l^{i+1} = g_l + K_l q_l^i \quad i = 1, \ldots, \nu,\ \nu \geq 2 \tag{3.24}$$

Residues: $$r_l^{\nu+1} = (q_l^{\nu+1} - g_l - K_l q_l^{\nu+1}) \tag{3.25}$$

Breakdown criterion: $$\rho_l^{\nu+1} = \left\| r_l^{\nu+1} \right\|_2, \ \frac{\rho_l^{\nu+1}}{\rho_0} < \varepsilon \quad \text{stop}$$

Coarse grid correction: $$d_l = r(q_l^{\nu+1} - g_l - K_l q_l^{\nu+1}) \tag{3.26}$$

$$\delta_{l-1} = (I - K_{l-1})^{-1} d_{l-1} \tag{3.27}$$

$$q_{l+1}^0 = q_l^{\nu+1} - P\delta_{l-1} \tag{3.28}$$

Here r is $n_l \times n_{l-1}$ restriction matrix and P is $n_{l-1} \times n_l$ prolongation matrix. The indices $l-1$ and l are used for the coarse grid and fine grid respectively.

Convergence Of The Two-Grid Method

The mapping $q_l^i \to q_l^{i+1}$ of the two-grid algorithm is affined an has the representation

$$q_l^{i+1} = M_l^{TGM} q_l^i + C_l \tag{3.29}$$

where M_l^{TGM} is the two-grid iteration matrix.

Lemma 3.1. The two-grid iteration matrix M_l^{TGM} has the form [7]

$$M_l^{TGM} = \left[I - P(I - K_{l-1})^{-1} r(I - K_l)\right] K_l \quad \text{for all } l \geq 1$$

The partition of this matrix yields

$$\begin{aligned} M_l^{TGM} &= \left\{(I - Pr) + P(I - K_{l-1})^{-1}\left[(I - K_{l-1})r - r(I - K_l)\right]\right\} K_l \\ &= \left\{(I - Pr) + P(I - K_{l-1})^{-1}\left[rK_l - K_{l-1}r\right]\right\} K_l \end{aligned} \tag{3.30}$$

A sufficient condition for the convergence of this method ist the validity of the contraction condition

$$\left\|M_l^{TGM}\right\|_{A_l} < 1 \tag{3.31}$$

where M_l^{TGM} is given in (3.30).

3.2.3. Multi-Gird Method

The multi-grid iteration consists of a smoothing step and a coarse grid correction. The latter step uses the restricted defect $r(q_l^{\nu+1} - g_l - K_l q_l^{\nu+1})$. The resulting iteration is defined by the following recursive procedure:

Smoothing step: $$q_l^{i+1} = g_l + K_l q_l^i \quad i = 1, \ldots, \nu,\ \nu \geq 2 \tag{3.32}$$

Residues: $$r_l^{\nu+1} = (q_l^{\nu+1} - g_l - K_l q_l^{\nu+1}) \tag{3.33}$$

Breakdown criterion: $$\rho_l^{\nu+1} = \left\|r_l^{\nu+1}\right\|_2, \ \frac{\rho_l^{\nu+1}}{\rho_0} < \varepsilon \quad \text{stop}$$

Coarse grid correction: $$d_{l-1} = r(q_l^{\nu+1} - g_l - K_l q_l^{\nu+1}) \tag{3.34}$$

Multi-grid approximation $$(\delta_{l-1} = d_{l-1} + K_{l-1}\delta_{l-1}) \tag{3.35}$$

$$q_{l+1}^0 = q_l^{\nu+1} - P\delta_{l-1}. \tag{3.36}$$

Convergence Of The Multi-Grid Method

The mapping $q_l^i \to q_l^{i+1}$ of the multi-grid algorithm has the representation [7]

$$q_l^{i+1} = M_l^{MGM} q_l^i + C_l \tag{3.37}$$

where M_l^{MGM} is the multi-grid iteration matrix.

Lemma 3.2. The multi-grid iteration matrix M_l^{MGM} has the form [7]

$$M_l^{MGM} = M_l^{TGM} + P(M_{l-1}^{MGM})^2(I - K_{l-1})^{-1}r(I - K_l)K_l. \tag{3.38}$$

An alternative representation to (3.38) is

$$M_l^{MGM} = M_l^{TGM} + P(M_{l-1}^{MGM})^2\left[r - (I - K_{l-1})^{-1}(rK_l - K_{l-1}r)\right]K_l. \tag{3.39}$$

A sufficient condition for the convergence of this method ist the validity of the contraction condition

$$\left\|M_l^{MGM}\right\|_{A_l} < 1 \tag{3.40}$$

where M_l^{MGM} is given in (3.39).

3.2.4. Conjugate Gradient Iteration

This is an iteration method for solving the linear system

$$C_l a_l = b_l \tag{3.41}$$

where $C_l = (A_l - B_l)$.
It is an effective method for symmetric and positive definite systems.
This CG-iteration is given by the following algorithm [6]:

1. Choose an initial vector a_l^0 and compute $r_0 = C_l a_l^0 - b_l$.
 Set $p_0 = r_0$ and $k = 0$

2. Compute

$$\begin{aligned} \alpha_k &= \frac{r_k^T p_k}{p_k^T C_l p_k} \\ a_l^{k+1} &= a_l^k + \alpha_k p_k \\ r_{k+1} &= C_l a_l^{k+1} \end{aligned}$$

3. Stop if $\dfrac{\|r_{k+1}\|_2}{\|r_k\|_2} < \varepsilon$

4. Compute

$$\begin{aligned} \beta_k &= \frac{r_{k+1}^T C_l p_k}{p_k^T C_l p_k} \\ p_{k+1} &= r_{k+1} + \beta_l p_k. \end{aligned}$$

Convergence Of The Conjugate Gradient Method

From [12] follows that

$$\varepsilon \cdot \langle q, q \rangle_{L^2(\Gamma)} \leq \langle Aq, q \rangle_{L^2(\Gamma)} \leq (2 - \varepsilon) \cdot \langle q, q \rangle_{L^2(\Gamma)} \tag{3.42}$$

where $A = (I - K)$.
Let $q \in H_l \subset L^2(\Gamma)$ then we define

$$q(t) = \sum_{k=1}^{n_l} q_k \Phi_k(t). \tag{3.43}$$

Set equation (3.43) into (3.42) we get

$$\varepsilon \cdot \left\| \sum_{k=1}^{n_l} q_k \Phi_k \right\|_{L^2(\Gamma)}^2 \leq \sum_{k,j}^{n_l} q_k q_j \langle A\Phi_k, \Phi_j \rangle_{L^2(\Gamma)} \leq (2 - \varepsilon) \cdot \left\| \sum_{k=1}^{n_l} q_k \Phi_k \right\|_{L^2(\Gamma)}^2 . \tag{3.44}$$

Now

$$\left\| \sum_{k=1}^{n_l} q_k \Phi_k \right\|_{L^2(\Gamma)}^2 = \int_0^1 \left| \sum_{k=1}^{n_l} q_k \Phi_k(t) \right|^2 dt = \sum_{k,j}^{n_l} q_k q_j \int_0^1 \Phi_k(t) \Phi_j(t) dt = (A_l q, q) \tag{3.45}$$

Substituting (3.45) into (3.44) yields

$$\varepsilon (A_l q, q) \leq (C_l q, q) \leq (2 - \varepsilon) \cdot (A_l q, q). \tag{3.46}$$

Lemma 3.3. Let λ_k be the real, positive eigenvalue of the mass matrix A_l, and it holds

$$\left| \frac{\lambda_{max}}{\lambda_{min}} \right| \leq c \tag{3.47}$$

then follows

$$\lambda_{min}(q, q)_{A_l} \leq (A_l q, q) \leq \lambda_{max}(q, q)_{A_l}. \tag{3.48}$$

Thus one obtains

$$\lambda_{min}\|q\|^2_{l^2(\Gamma)} \leq (A_l q, q) \leq \lambda_{max}\|q\|^2_{l^2(\Gamma)}. \tag{3.49}$$

From (3.46) follows immediately

$$(\varepsilon)\lambda_{min}\|q\|^2_{l^2(\Gamma)} \leq (C_l q, q)_{A_l} \leq (2-\varepsilon)\cdot\lambda_{max}\|q\|^2_{l^2(\Gamma)}. \tag{3.50}$$

The condition number $\kappa(C_l)$ of the matrix C_l can then be estimated to give

$$\kappa(C_l) \leq \frac{(2-\varepsilon)\cdot\lambda_{max}}{(\varepsilon)\cdot\lambda_{min}} \tag{3.51}$$

using (3.47) then follows

$$\kappa(C_l) \leq c\cdot\frac{(2-\varepsilon)}{(\varepsilon)}\ .$$

Theorem 3.1. For a positive matrix C_l converges the Conjugate Gradient iteration with the convergence estimation [8]

$$\left\|e^k\right\|_{C_l} \leq 2\left(\frac{(\kappa(C_l)-1)^{\frac{1}{2}}}{(\kappa(C_l)+1)^{\frac{1}{2}}}\right)^k \left\|e^0\right\|_{C_l} \tag{3.52}$$

where $\left\|e^k\right\|_{C_l} = \left\|a_l^k - a_l\right\|_{C_l}$ and $\left\|e^0\right\|_{C_l} = \left\|a_l^0 - a_l\right\|_{C_l}$.

3.3. Computation Of The Visibility Function $\beta(x, y)$

We illustrate in the following steps the method for which how the visibility function $\beta(x, y)$ can be computed (see Fig.4)

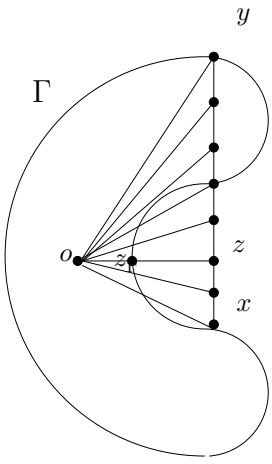


Fig.4

- We define

 G : be the straight segment between the points x and y

 $G := \{z \in \mathbb{R}^2 : z = x + \varphi(x + y), \varphi \in [0, 1]\}$.

 Question (1): is $G \subset \Omega$?

- We define

 $\widetilde{G}$: be the set of points such that

 $\widetilde{G} := \left\{z_i : z_i = x + \varphi_i \cdot (y - x), \varphi_i = \frac{i-1}{|x-y|}, i = 1, \ldots, m, m \in \mathbb{N}\right\}$

 $\widetilde{G}$ is thus an approximation of the line G.

 Question (2): is $\widetilde{G} \subset \Omega$?

 For all $z \in \widetilde{G}$

- We require the point 0 to be always situated in the region Ω,

- We determine next z_Γ, and

- Prove then if $|z_\Gamma| < |z|$

If this is the case then follows immediately that $\beta(x, y) = 0$

Question (3): How can z_Γ be determined ?

First we set $z_\Gamma = \alpha z, \alpha \in \mathbb{R}$.

The determination of α is necessary, therefore we demand

- $z_\Gamma \in \Gamma$ (see Fig.5)

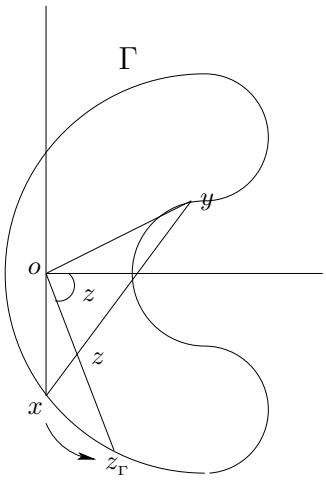


Fig.5

- $\arg z_\Gamma = \arg z$

To satisfy the first requirement, we set $x = X(t_0)$ and define $\Gamma = \{x = X(t), t \in [0,1]\}$. Determine next $t_1 = t_0 + \varepsilon$:
When $z_\Gamma = X(t_1)$, follows immediately the first requirement.

4. THE ASYMPTOTIC ERROR ANALYSIS

4.1. Theoretical Error Estimation

Most the asymptotic error estimates $\|q - q_h\|_{L^2(\Gamma)}$ are formulated in Sobolev spaces. It holds the following lemma

Lemma 4.1. (Cea's Lemma [10, 13])
The integral operator $A = I - K$ is a pseudodifferential operator of order zero. Therefore follows that for all $q \in L^2(\Gamma)$ the quasi-optimal error estimates

$$\|q - q_h\|_{L^2(\Gamma)} \leq c \inf_{w_h \in H_h} \|q - w_h\|_{L^2(\Gamma)} \tag{4.1}$$

holds, where the constant c is independent of h and q.

Theorem 4.1. The integral operator A is a strongly elliptic pseudodifferential operator of order α. Further holds for the two dimensional case

$$\alpha < 2r + 1 \tag{4.2}$$

Let $\alpha - d \leq \sigma \leq \frac{\alpha}{2} \leq \tau \leq d$ be satisfied and in addition let q_h be the Galerkin solution of the Galerkin equation

$$\langle Aq_h, w_h \rangle_{L^2(\Gamma)} = \langle g, w_h \rangle_{L^2(\Gamma)} \qquad \text{for all } w_h \in H_h$$

then we have the asymptotic error estimate

$$\|q_h - q\|_{H^\sigma(\Gamma)} \leq ch^{\tau-\sigma} \|q\|_{H^\tau(\Gamma)} \ . \tag{4.3}$$

Lemma 4.2. Let the ansatz functions be piecewise linear. Moreover $(I - K)$ is a pseudodifferential operator of order $\alpha = 0$, then follows from (4.3) the error estimate

$$\|q_h - q\|_{L^2(\Gamma)} \leq ch^2 \|q\|_{H^2(\Gamma)} . \tag{4.4}$$

For the boundary method one needs to compute numerically the coefficients $B_{l,k}$ of the view factor matrix B. Its computation is carried out by a suitable form of numerical integration. If the numerical integration is not accurately carried out then one expects quite high integration error. The accuracy of the numerical integration must be discussed in relation to the asymptotic error estimation therefore it is necessary to consider the following Lemma from Strang [4].

Theorem 4.2. (Strang Lemma [4])
We consider a family of approximated bilinear forms a_h which are uniformly H_h-elliptic. Then there exists a constant c that is independent of q and h and it holds the following inequality

$$\begin{aligned} \|q - q_h\|_{L^2(\Gamma)} \ \leq \ & c \left(\inf_{w_h \in H_h} \left\{ \|q - w_h\|_{L^2(\Gamma)} + \sup_{w_h \in H_h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{L^2(\Gamma)}} \right\} \right. \\ & \left. + \sup_{w_h \in H_h} \frac{|g(w_h) - g_h(w_h)|}{\|w_h\|_{L^2(\Gamma)}} \right) \end{aligned} \tag{4.5}$$

where the terms $a(v_h, w_h)$, $g(w_h)$, $g_h(w_h)$ and $a_h(v_h, w_h)$ in (4.5) are defined as follows

$a(v_h, w_h) = \langle (I-K)v_h, w_h \rangle_{L^2(\Gamma)} = \langle Av_h, w_h \rangle_{L^2(\Gamma)}$

$g(w_h) = \langle g, w_h \rangle_{L^2(\Gamma)}$

$g_h(w_h) = \langle g_h, w_h \rangle_{L^2(\Gamma)}$

and

$a_h(v_h, w_h) = \langle a_h, w_h \rangle_{L^2(\Gamma)}$.

The approximation $a_h(v_h, w_h)$ has the form

$$\begin{aligned} a_h(v_h, w_h) &= \int_\Gamma (I-K)v_h w_h d\Gamma_x = \int_\Gamma v_h(x) w_h(x) d\Gamma_x \\ &\quad - \int_\Gamma \int_\Gamma (1-\varepsilon(x)) G(x,y) v_h(x) w_h(y) d\Gamma_x d\Gamma_y \end{aligned}$$

The coefficients $A_{k,l}$ of the mass matrix A (without the Quadrature error) are

$$A_{k,l} = a(\Phi_k, \Phi_l) = \sum_{k=1}^{n} \left\{ \int_\Gamma \Phi_k(x)\Phi_l(x) d\Gamma_x - \int_\Gamma \int_\Gamma (1-\varepsilon(x)) G(x,y) \Phi_k(x) \Phi_l(y) d\Gamma_x d\Gamma_y \right\}.$$

If we now replace the above integration by Gaussian quadrature. This yields the following approximation formula

$$\widetilde{A}_{k,l} = a_h(\Phi_k, \Phi_l) = \sum_{i=1}^{m} W_i F_{k,l}(x_i) + \sum_{i=1}^{m} \sum_{j=1}^{m} W_i W_j E_{k,l}(x_i, y_j),$$

where $F_{k,l}$ and $E_{k,l}$ are given by

$$F_{k,l} = \Phi_k(x)\Phi_l(x),$$

and

$$E_{k,l}(x,y) = (1-\varepsilon(x)) G(x,y) \Phi_k(x) \Phi_l(y)$$

here m denotes the order of the quadrature and the coefficients W_i and W_j are the weights of the quadrature form.

The ellipticity of a_h follows directly from Lemma 2.8 [12].

It holds

$$\varepsilon \|q\|^2_{L^2(\Gamma)} \leq \langle Aq, q \rangle_{L^2(\Gamma)} \leq (2-\varepsilon) \|q\|^2_{L^2(\Gamma)} \tag{4.6}$$

Let the approximation operator A_h satisfies the approximation inequality

$$\|(A - A_h)q\|_{L^2(\Gamma)} \leq c h_l^\tau \|q\|_{H^\tau(\Gamma)} \tag{4.7}$$

where τ is defined in (3.5).
Let q_h be our assigned ansatz function, then follows

$$\varepsilon\|q_h\|^2_{L^2(\Gamma)} \leq \langle Aq_h, q_h\rangle_{L^2(\Gamma)} + ch_l^{\tau}\|q_h\|_{H^{\tau}(\Gamma)} \cdot \|q_h\|_{L^2(\Gamma)} \tag{4.8}$$

with the help of the inverse inequality (3.7) we get

$$\varepsilon\|q_h\|^2_{L^2(\Gamma)} \leq \langle Aq_h, q_h\rangle_{L^2(\Gamma)} + c_1^* \left(\frac{h_l}{h_{l-1}}\right)^{\tau} \|q_h\|^2_{L^2(\Gamma)}. \tag{4.9}$$

Finally we obtain

$$\left(\varepsilon - c_1^* \left(\frac{h_l}{h_{l-1}}\right)^{\tau}\right) \|q_h\|^2_{L^2(\Gamma)} \leq \langle A_h q_h, q_h\rangle_{L^2(\Gamma)} \tag{4.10}$$

under the assumption $c_2^* \leq \left(\frac{h_l}{h_{l-1}}\right) \leq c_3^*$ one obtains for the case $\tau = 1$

$$\langle A_h q_h, q_h\rangle_{L^2(\Gamma)} \geq \frac{1}{2}\varepsilon \cdot \|q_h\|^2_{L^2(\Gamma)}. \tag{4.11}$$

Hence ellipticity is proved. From this condition follows how exact the numerical quadrature error must be.

5. NUMERICAL RESULTS

5.1. Numerical Examples For The Solution Of The System Of Equations

Since the convergence requirements of the four solution methods are satisfied [12], then we can apply now these methods to solve the following two-dimensional convex and non-convex enclosures.

Convex Enclosure

Example 5.1. Let Ω be the domain of an ellipse. The boundary of this ellipse has the following parameterization

$$Gamma = \left\{ x \in \mathbb{R}^2 : x = \begin{pmatrix} a\cos 2\pi t \\ b\sin 2\pi t \end{pmatrix}, a = 4, b = 2, 0 \leq t < 1 \right\}. \tag{5.1}$$

The computation of the coefficients $A_{l,k} = \langle \Phi_{k,n}, \Phi_{l,n} \rangle$, $b_n = \langle g, \Phi_{l,n} \rangle$ and $B_{l,k} = \langle K\Phi_{k,n}, \Phi_{l,n} \rangle$ have been carried out be Gaussian quadrature form.
Here we have $g(t) = \varepsilon(t)\sigma T^4(t)$ with
The emissivity coefficient $\varepsilon = 0.9$
The Boltzmann coefficient $\sigma = 5.6696 \times 10^{-8}$ and
The surface temperature $T(t) = \frac{1}{2}(T_1 + T_2) - \frac{1}{2}(T_2 - T_1)\cos 2\pi t$, where $T_1 = 1000$ and $T_2 = 1800$.
Table (I) shows the numerical results for the solutions of equation (3.20) by using Picard's iteration, two-grid and multi-grid methods and CG-iteration method for the ellipse. It contains both the number of iteration steps and the required CPU-time in second. The mesh width $h_l = \frac{1}{n_l}$ with $n_l = 2^l$. The number $n = n_l$ denotes the parameter of the solved problem. The four iteration methods converge for all levels l. Comparing these iterations together we see clearly that the two-grid and multi-grid methods require both a small number of iterations and CPU-time in comparison with the Picard's iteration. On the other hand the CG-iteration needs more iteration steps but less CPU-time in comparison with the other methods.

Table I. Solution Methods for an Ellipse

n_l	Picard		Two-grid		Multi-grid		CG	
	Iter	sec	Iter	sec	Iter	sec	Iter	sec
32	14	< 1	6	< 1	2	< 1	16	< 1
64	14	0.50	6	< 1	2	< 1	18	< 1
128	14	2.02	6	1.12	2	< 1	19	< 1
256	14	8.05	6	4.42	2	1.51	20	0.51
512	14	32.09	6	16.69	2	6.01	20	2.05
1024	14	128.26	6	69.98	2	24.07	20	8.16

Non-Convex Enclosure

Example 5.2. We consider for an example the non-convex curve shown in Fig.6. In this case the visibility function $\beta(t, \tau)$ must be taken into consideration, with $\beta(t, \tau)$ is defined in (2.18). The computation of this visibility function has been

illustrated in section 3.3. Table (II) contains the numerical resuts for this non-convex case.

Table II. Solution Methods for the Nonconvex Curve in Fig.6

n_l	Picard		Two-grid		Multi-grid		CG	
	Iter	sec	Iter	sec	Iter	sec	Iter	sec
32	16	< 1	8	< 1	3	< 1	27	< 1
64	16	0.58	8	< 1	3	< 1	31	< 1
128	16	2.31	8	1.38	3	0.53	40	< 1
256	16	9.14	8	5.49	3	2.09	43	1.07
512	16	36.50	8	21.98	3	8.31	43	4.23
1024	16	145.48	8	86.92	3	33.02	43	16.88

5.2. Numerical Examples For The Error Estimation

5.2.1. Convex Case

a) Γ Describes the boundary of a circle

Example 5.3. Let $q(t) = \cos 2\pi t$ for $0 \le t \le 1$ be the exact solution of the integral equation

$$q(t) = g(t) + (1 - \varepsilon) \int_0^1 G^*(t, \tau) q(\tau) |\dot{x}(\tau)| d\tau \ . \tag{5.2}$$

Then the exact $g(t)$ for the given exact $q(t)$ has been calculated as follows
For the unit circle $\vec{n}(\tau) \cdot (\vec{t} - \vec{\tau}) = \frac{1}{2}$ and $|\vec{\tau} - \vec{t}| = 2|\sin \pi(t - \tau)|$.
Then the kernel $G^*(t, \tau)$ in (2.17) reduced to

$$G^*(t, \tau) = \frac{1}{2} \cdot \frac{1}{4} |\vec{t} - \vec{\tau}| = \frac{1}{4} |\sin \pi(t - \tau)|. \tag{5.3}$$

Substituting (5.3) into (5.2) with $|\dot{x}(\tau)| = \cos 2\pi t$ to obtain the exact $g(t)$ as

$$g(t) = \cos 2\pi t + \frac{1}{3}(1 - \varepsilon(t)) \cos 2\pi t \tag{5.4}$$

This computed exact $g(t)$ in (5.4) has then been used in our program to obtain the approximat solution q_h with the help of our numerical iterations.

b) Γ Describes the boundary of square

Example 5.4. Let $q(t) = x_1(t)$ be the exact solution for the case of a unit square. Then the exact $g(t)$ can be computed as follows:
For $t \geq 0$ and $t < 0.25$ we have

$$\begin{aligned} g_1(t) &= 4t - 4(1-\varepsilon(t)) \left\{ \int_0^{1/4} G_{11}^*(t-\tau)\cdot 4\tau d\tau + \int_{1/4}^{1/2} G_{12}^*(t,\tau)\cdot 1 d\tau \right. \\ &\quad \left. + \int_{1/2}^{3/4} G_{13}^*(t-\tau)\cdot(3-4\tau)d\tau + \int_{3/4}^{1} G_{14}^*(t,\tau)\cdot 0 d\tau \right\}, \end{aligned} \tag{5.5}$$

where
$G_{11}^* = 0, \qquad G_{12}^* = G_{21}^* = \dfrac{(1-4t)(4\tau-1)}{2\left[(4t-1)^2+(4\tau-1)^2\right]^{2/3}},$
$G_{13}^* = G_{31}^* = \dfrac{1}{2\left[16(t-\frac{3}{4}+\tau)^2+1\right]^{2/3}}$ and
$G_{14}^* = G_{41}^* = \dfrac{16t}{2\left[16t^2+16(1-\tau)^2+1\right]^{2/3}}$

For $t \geq 0.25$ and $t < 0.5$ we have

$$\begin{aligned} g_2(t) &= 1.0 - 4(1-\varepsilon(t)) \left\{ \int_0^{1/4} G_{21}^*(t,\tau)\cdot 4 d\tau + \int_{1/4}^{1/2} G_{22}^*(t,\tau)\cdot 1 d\tau \right. \\ &\quad \left. + \int_{1/2}^{3/4} G_{23}^*(t,\tau)\cdot(3-4\tau)d\tau + \int_{3/4}^{1} G_{24}^*(t,\tau)\cdot 0 d\tau \right\}, \end{aligned} \tag{5.6}$$

with $G_{22}^* = 0, \qquad G_{23}^* = G_{32}^* = \dfrac{(1-2t)(2\tau-1)}{2\left[(2t-1)^2+(2\tau-1)^2\right]^{2/3}},$
and $G_{24}^* = G_{42}^* = \dfrac{1}{2\left[16(t-\frac{5}{4}+\tau)^2+1\right]^{2/3}}.$

For $t \geq 0.5$ and $t < 0.75$ we have

$$\begin{aligned} g_3(t) &= (3-4t) - 4(1-\varepsilon(t)) \left\{ \int_0^{1/4} G_{31}^*(t,\tau)\cdot 4\tau d\tau + \int_{1/4}^{1/2} G_{32}^*(t,\tau)\cdot 1 d\tau \right. \\ &\quad \left. + \int_{1/2}^{3/4} G_{33}^*(t,\tau)\cdot(3-4\tau)d\tau + \int_{3/4}^{1} G_{34}^*(t,\tau)\cdot 0 d\tau \right\}, \end{aligned} \tag{5.7}$$

where $G^*_{33} = 0$ and $G^*_{34} = G^*_{43} = \dfrac{(3-4t)(4\tau-3)}{2\left[(3-4t)^2+(4\tau-3)^2\right]^{2/3}}$.

For $t \geq 0.75$ and $t < 1.0$ holds

$$g_4(t) = -4(1-\varepsilon(t))\left\{\int_0^{1/4} G^*_{41}(t,\tau)\cdot 4\tau d\tau + \int_{1/4}^{1/2} G^*_{42}(t,\tau)\cdot 1 d\tau + \int_{1/2}^{3/4} G^*_{43}(t,\tau)\cdot(3-4\tau)d\tau + \int_{3/4}^{1} G^*_{44}(t,\tau)\cdot 0 d\tau\right\}, \qquad (5.8)$$

where $G^*_{44} = 0$.

The exact $g(t)$ in (5.5), (5.6), (5.7) and (5.8) has been explicity calculated. Tables (III) and (IV) contain the numerical results for the two computed $g(t)$ in (5.4) and (5.5 − 5.8) respectively. They show the L_2-error $\|q-q_h\|_{L^2}$ and the order of convergence. In this case we obtain an error estimation for $\|q-q_h\|_{L^2}$ of order $O(h^2)$.

We conlude that the theoretical error estimation (4.4) and the numerical results in tables (III) and (IV) are equivalent.

Table III. Error Estimation

Theoretical Value = 2.0

l	n_l	L_2-Error	Conv. Ord
2	4	8.505×10^{-2}	
3	8	1.747×10^{-2}	2.28
4	16	4.139×10^{-3}	2.08
5	32	1.021×10^{-3}	2.02
6	64	2.536×10^{-4}	2.01
7	128	6.534×10^{-5}	2.00
8	256	1.588×10^{-5}	2.00
9	512	3.976×10^{-6}	2.00

Table IV. Error Estimation

Theoretical Value = 2.0

l	n_l	L_2-Error	Conv. Ord
2	4	4.457×10^{-2}	
3	8	9.483×10^{-3}	2.23
4	16	2.207×10^{-3}	2.10
5	32	5.490×10^{-4}	2.01
6	64	1.369×10^{-4}	2.01
7	128	3.414×10^{-5}	2.00
8	256	5.537×10^{-6}	2.00
9	512	2.139×10^{-6}	2.00

5.2.2. Non-Convex Case

Example 5.5. Let

$$q(t) = 1 + \begin{cases} t^2(t-\frac{1}{4})^2 & \text{for } t \in [0, \frac{1}{4}) \\ (t-\frac{1}{4})^2(t-\frac{1}{2})^2 & \text{for } t \in [\frac{1}{4}, \frac{1}{2}) \\ (t-\frac{1}{2})^2(t-\frac{3}{4})^2 & \text{for } t \in [\frac{1}{2}, \frac{3}{4}) \\ (t-\frac{3}{4})^2(t-1)^2 & \text{for } t \in [\frac{3}{4}, 1) \end{cases} \tag{5.9}$$

be the exact solution for the non-convex curve (see Fig.6),

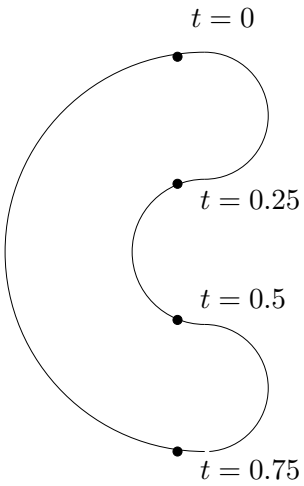


Fig.6

then the exact $g(t)$ can be calculated from the integral equation (2.17) as follows: For $t \geq 0$ and $t < 0.5$ we have

$$g_1(t) = q_1(t) - (1 - \varepsilon(t)) \left(\int_0^{1/4} G_1(t, \tau) \cdot q_1(\tau) \cdot 4\pi d\tau \right) \tag{5.10}$$

where $q_1(t) = 1 + t^2(t - \frac{1}{4})^2$ and $G_1(t, \tau) = \frac{1}{4} |\sin 2\pi(t - \tau)|$.

For $t \geq 0.25$ and $t < 0.5$ we have

$$g_2(t) = q_2(t) - (1 - \varepsilon(t)) \left(\int_{1/4}^{1/2} G_2(t, \tau) \cdot q_2(\tau) \cdot 4\pi d\tau \right) \tag{5.11}$$

where $q_2(t) = 1 + (t - \frac{1}{4})^2(t - \frac{1}{2})^2$ and $G_2(t, \tau) = \frac{1}{4} |\sin 2\pi(t - \tau)|$.

For $t \geq 0.5$ and $t < 0.75$ we have

$$g_3(t) = q_3(t) - (1 - \varepsilon(t)) \left(\int_{1/2}^{3/4} G_3(t, \tau) \cdot q_3(\tau) \cdot 4\pi d\tau \right) \tag{5.12}$$

where $q_3(t) = 1 + (t - \frac{1}{2})^2(t - \frac{3}{4})^2$ and $G_3(t, \tau) = \frac{1}{4} |\sin 2\pi(t - \tau)|$.

For $t \geq 0.75$ and $t < 1.0$ we have

$$g_4(t) = q_4(t) - (1 - \varepsilon(t)) \left(\int_{3/4}^{1} G_4(t, \tau) \cdot q_4(\tau) \cdot 12\pi d\tau \right) \tag{5.13}$$

where $q_4(t) = 1 + (t - \frac{3}{4})^2(t - 1)^2$ and $G_4(t, \tau) = \frac{1}{12} |\sin 2\pi(t - \tau)|$.

The exact $g(t)$ in (5.10), (5.11), (5.12) and (5.13) has been explicity computed. Table (V) contains the numerical results for this computed exact $g(t)$ (5.10 − 5.13). The table shows the L_2-error $\|q - q_h\|_{L^2}$ and the order of convergence. We see clearly that the L_2-error $\|q - q_h\|_{L^2}$ for this non-convex case is of order $O(h^2)$. We finally conclude that the theoretical error estimation (4.4) for $\|q - q_h\|_{L^2}$ and the numerical results in table (V) for the non-convex case are in good agreement.

Table V. Error Estimation

Theoretical Value = 2.0

l	n_l	L_2-Error	Conv. Ord
2	4	1.2345×10^{-1}	
3	8	2.6834×10^{-2}	2.16
4	16	6.3138×10^{-3}	2.09
5	32	1.5437×10^{-3}	2.03
6	64	3.8210×10^{-4}	2.01
7	128	9.5049×10^{-5}	2.01
8	256	2.3760×10^{-5}	2.00
9	512	5.9400×10^{-6}	2.00

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ABSTRACT. The aim of this paper is to study the Blackman-type sampling series. In Signal Analysis the Blackman window has been used over 40 years [1], [4]. Main goal of this paper is to present a mathematical treatment of approximation problems by the Blackman-type sampling series. We considered cases when we have a very good order of approximation. In some cases we are able to compute exact values of those operator norms.

Keywords: Blackman window function, Blackman kernel, operator norms, order of approximation.

1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of all naturals, all integers, all real and all complex numbers, respectively. Let $C(\mathbb{R})$ be the space of all uniformly continuous and bounded functions $f : \mathbb{R}\to\mathbb{R}$ (or $\mathbb{C}$) endowed with the supremum norm $\|\cdot\|_C$. Let $L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ be the space of all measurable functions f on $\mathbb{R}$ for which the norm

$$\|f\|_p := \left\{ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(t)|^p dt \right\}^{1/p},$$

$$\|f\|_\infty := \operatorname{ess\,sup} \{ |f(t)| : t \in \mathbb{R} \}$$

is finite. For $\sigma \geqslant 0$ and $1 \leqslant p \leqslant \infty$ let B_σ^p be the class of those bounded functions $f \in L^p(\mathbb{R})$ which can be extended to an entire function $f(z)$ ($z \in \mathbb{C}$) of exponential type σ ([3] or [10], 4.3.1), i. e.,

$$|f(z)| \leqslant e^{\sigma|y|} \|f\|_C \quad (z = x + iy \in \mathbb{C}).$$

The Fourier transform $f^\wedge$ of $f \in L(\mathbb{R})$ is defined for $v \in \mathbb{R}$ by

$$f^\wedge(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ivt} dt.$$

In [3] (and references cited there), P.L.Butzer and his school have considered the generalized sampling series given by ($t \in \mathbb{R}$; $W > 0$)

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) s(Wt-k) \tag{1}$$

for a $f \in C(\mathbb{R})$. There it was shown that the equality

$$\lim_{W\to\infty} (S_W f)(t) = f(t),$$

uniformly on $\mathbb{R}$, is essentially equivalent to each of the following two assertions

$$(i) \sum_{k=-\infty}^{\infty} s(x-k) = 1, \quad x \in [0,1);$$

$$(ii)\ s^\wedge(2k\pi) = 0,\ k \in \mathbb{Z}\setminus\{0\};\ s^\wedge(0) = (2\pi)^{-1/2}.$$

The well known Whittaker-Kotelnikov-Shannon sampling series is defined by the kernel $s(x) = \operatorname{sinc}(x) := \sin\pi x/(\pi x)$. Let us introduce a band-limited kernel s defined via a window function $\lambda \in C_{[0,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geqslant 1$) by equality

$$s(t) := \int_0^1 \lambda(u)\cos(\pi t u)\, du. \tag{2}$$

A lot of kernels are defined by (2) , e.g.

1) $\lambda(u) = 1$ defines the sinc function,

2) $\lambda(u) = 1-u$ defines the Fejér kernel $s_F(t) = \frac{1}{2}\operatorname{sinc}^2\frac{t}{2}$,

3) $\lambda_j(u) = \cos\pi(j+1/2)u$, $j = 0,1,2,\ldots$ defines the Rogosinski-type kernel (see [7], [6]) in the form

$$r_j(t) := \frac{1}{2}[\operatorname{sinc}(t+j+1/2) + \operatorname{sinc}(t-j-1/2)] \tag{3}$$

$$= \frac{(-1)^j}{\pi}\frac{(j+1/2)\cos\pi t}{(j+1/2)^2 - t^2}. \tag{4}$$

Let us recall some auxiliary results.

It is known ([3],[5]), that in (2) the kernel $s \in B^1_\pi$ (i.e the kernel is band-limited) and for $f \in C(\mathbb{R})$ we have $S_W f \in B^\infty_{\pi W}$. We need the classical sampling theorem ([3], Th. 6.3a): for $g \in B^\infty_\sigma$ with $\sigma < \pi W$ we have

$$g(t) = \sum_{k=-\infty}^{\infty} g(\frac{k}{W})\operatorname{sinc}(Wt-k) =: (S^{\operatorname{sinc}}_W g)(t) \tag{5}$$

and if $\sigma = \pi W$ this is not valid.

Below let us denote the sampling series (5) by $S^{\operatorname{sinc}}_W g$.

Some auxiliary facts from the approximation theory are needed. For $f \in C(\mathbb{R})$ and $\delta \geqslant 0$ the k-th modulus of continuity ([2], p.76) is defined by

$$\omega_k(f,\delta) := \sup_{|h|\leqslant\delta} \|\Delta^k_h f(\cdot)\|_C,$$

where

$$\Delta_h^k f(x) = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} f(x+lh). \tag{6}$$

The modulus of continuity has the following properties ([2], p. 76; [10], 3.3):

$$\begin{array}{ll} \omega_k(f,\delta) \leqslant 2^{k-r}\omega_r(f,\delta) & \text{for any } r \in \mathbb{N},\ r \leqslant k, \\ \omega_k(f, j\delta) \leqslant j^k \omega_k(f,\delta) & \text{for any } j \in \mathbb{N}, \\ \omega_k(f,\lambda\delta) \leqslant (1+\lambda)^k \omega_k(f,\delta) & \text{for any } \lambda > 0, \\ \omega_k(f,\delta) \leqslant \delta^k \|f^{(k)}\|_C & \text{for any } f^{(k)} \in C(\mathbb{R}). \end{array} \tag{7}$$

We need a special Jackson-type inequality (cf. [10], 8.7, Problem 23 or [9], Lemma 2).

Proposition 1. *For given $f \in C(\mathbb{R})$ there exist $g^* \in B_\sigma^\infty$ and $M_k > 0$ $(k \in \mathbb{N})$ such that for every $\sigma \geqslant 2$*

$$\|f - g^*\|_C \leqslant M_k \omega_k(f, \frac{1}{\sigma}).$$

The aim of this paper is to study the generalized sampling series (1) defined by the Blackman-type window functions. In Signal Analysis the Blackman window

$$\lambda_B(u) = 0.42 + \frac{1}{2}\cos \pi u + 0.08 \cos 2\pi u.$$

has been used in many situations (see [1],[4] and references cited there). Main goal of this paper is to present a mathematical treatment of approximation problems by the Blackman sampling series. We shall consider a family of windows

$$\lambda_{B,a}(u) := a + \frac{1}{2}\cos \pi u + (\frac{1}{2} - a)\cos 2\pi u \quad (a \in \mathbb{R}), \tag{8}$$

which gives by (2) for sampling series a kernel

$$s_{B,a}(t) := \int_0^1 \lambda_{B,a}(u) \cos(\pi u t) du. \tag{9}$$

Among other results we proved that the value of parameter $a = 27/64 = 0.4218\ldots$ gives a very small value of the norm of sampling operator $B_{W,27/64} : C(\mathbb{R}) \to C(\mathbb{R})$, but a very good order of approximation we have in case $a = 5/8$. For finding exact values of norms we used computer-aided (Mathematica) hypotheses, but all results can be proved analytically too.

The Blackman window is a special case of the general cosine window in the form (here and in following formulas $[x]$ is entire part of $x \in \mathbb{R}$)

$$\lambda_m(u) = \sum_{k=0}^{m} c_k \cos k\pi u = \sum_{k=0}^{\left[\frac{m}{2}\right]} c_{2k} \cos(2k\pi u) + \sum_{k=1}^{\left[\frac{m+1}{2}\right]} c_{2k-1} \cos((2k-1)\pi u).$$

For λ_m to be a window function we require that $\lambda_m(0) = 1$ and $\lambda_m(1) = 0$. Then we must have

$$\sum_{k=0}^{\left[\frac{m}{2}\right]} c_{2k} = \sum_{k=1}^{\left[\frac{m+1}{2}\right]} c_{2k-1} = \frac{1}{2},$$

which for $m=2$ gives $c_1=1/2$ and $c_0+c_2=1/2$. Therefore, from the mathematical point of view, the most general Blackman window can be represented in the form (8). The family of windows $\lambda_{B,a}$ in (8) contains also the Hann window (see [4]), if take $a=1/2$.

2. EXACT VALUES OF NORMS OF SOME BLACKMAN-TYPE SAMPLING OPERATORS

The kernel (9) has representations

$$\begin{aligned} s_{B,a}(t) &= \frac{[(3-8a)t^2+8a]\operatorname{sinc} t}{2(1-t^2)(4-t^2)} && (10)\\ &= a\operatorname{sinc}(t)+\frac{1}{4}[\operatorname{sinc}(t-1)+\operatorname{sinc}(t+1)] \\ &+ \frac{1-2a}{4}[\operatorname{sinc}(t-2)+\operatorname{sinc}(t+2)]. && (11)\end{aligned}$$

We want to compute the norm of the Blackman sampling operators $B_{W,a}: C(\mathbb{R})\to C(\mathbb{R})$ using equation ([3], Th.4.1)

$$\|B_{W,a}\| = \sup_{0\leqslant u\leqslant 1}\sum_{k=-\infty}^{\infty}|s_{B,a}(u-k)|. \tag{12}$$

In [8] we proved that the Hann sampling operator $H_W: C(\mathbb{R})\to C(\mathbb{R})$ has the norm

$$\|H_W\| = \|B_{W,1/2}\| = \frac{10}{3\pi}.$$

By the representation (10) we could claim that an easy case to compute norms $\|B_{W,a}\|$ might be when $0\leqslant a\leqslant 3/8$, since in (10) $(3-8a)t^2+8a\geqslant 0$ and then we are able to consider signs of $s_{B,a}$.

All computations of exact values of norms of sampling operators are based on the following steps. First, we identify the signs of $s_{B,a}(u-k)$ in (12) depending on $k\in\mathbb{Z}$ and u, which belongs to an interval with length one (keeping in mind, that the function $u\mapsto \sum\limits_{k=-\infty}^{\infty}|s_{B,a}(u-k)|$ is 1-periodic). Secondly, we split the series in (12) into parts (mainly into two parts) in such way as stated by the first step, i.e. we must be able to find closed expressions for all parts of our partition. Thirdly, the value of the operator norm is an extreme value of a function given in closed-form in second step. For reasonable hypotheses in steps two and three we used computer-package Mathematica.

We start with boundary cases $a=0$ or $a=3/8$ as the following theorem states.

Theorem 1. *We have*

$$\|B_{W,0}\| = \frac{362}{105\pi} = 1.0974..., \quad \|B_{W,3/8}\| = \frac{332}{105\pi} = 1.0064...$$

Proof. By (12) we have to consider the quantity

$$N_a(u) := \sum_{k=-\infty}^{\infty}|s_{B,a}(u-k)|, \tag{13}$$

where by 1-periodicity we will consider the arguments $u \in (0,1)$ only. We have by (10) for $0 \leqslant a \leqslant 3/8$

$$\operatorname{sgn} s_{B,a}(u-k) = \begin{cases} 1, & |k| \leqslant 2, \\ (-1)^{k+1}\operatorname{sgn} k, & |k| \geqslant 3. \end{cases}$$

Therefore, by (13)

$$\begin{aligned} N_a(u) &= \sum_{k=1}^{\infty}(|s_{B,a}(u-k)| + |s_{B,a}(u+k-1)|) \\ &= \sum_{k=1}^{3}[s_{B,a}(u-k) + s_{B,a}(u+k-1)] \\ &+ \sum_{k=4}^{\infty}(-1)^{k+1}[s_{B,a}(u-k) + s_{B,a}(u+k-1)] \\ &=: N_{a,1}(u) + N_{a,2}(u). \end{aligned} \tag{14}$$

Using the Rogosinski kernel (3) we can represent (11) in another form

$$s_{B,a}(t) = a[r_0(t-\frac{1}{2}) + r_0(t+\frac{1}{2})] + \frac{1-2a}{2}[r_0(t-\frac{3}{2}) + r_0(t+\frac{3}{2})]. \tag{15}$$

Denote ($k \in \mathbb{Z}$)

$$\begin{aligned} D_k(u) &:= r_0(u-k-\frac{1}{2}) + r_0(u+k-\frac{1}{2}) \\ &= \begin{cases} 2r_0(u-\frac{1}{2}), & k=0, \\ r_{k-1}(u-\frac{1}{2}) + r_k(u-\frac{1}{2}), & k \in \mathbb{N}, \end{cases} \end{aligned} \tag{16}$$

hence, by (15)

$$\begin{aligned} s_{B,a}(u-k) + s_{B,a}(u+k-1) &= a[D_k(u) + D_{k-1}(u)] \\ &+ \frac{1-2a}{2}[D_{k+1}(u) + D_{k-2}(u)]. \end{aligned} \tag{17}$$

First, let consider $N_{a,2}$ in (14). Since by (10) $s_{B,a}(t) = O(t^{-3})$ as $t \to \infty$, the series $N_{a,2}(u)$ in (14) is convergent. Therefore, we consider the partial sums of $N_{a,2}(u)$ in form

$$\begin{aligned} &\sum_{k=4}^{4n+3} (-1)^{k-1}[s_{B,a}(u-k) + s_{B,a}(u+k-1)] \\ &= \sum_{k=4}^{4n+3}(-1)^{k-1}\left(a[D_k(u) + D_{k-1}(u)] + \frac{1-2a}{2}[D_{k+1}(u) + D_{k-2}(u)]\right) \\ &= a\sum_{k=2}^{2n+1}[-D_{2k}(u) - D_{2k-1}(u) + D_{2k+1}(u) + D_{2k}(u)] \\ &+ \frac{1-2a}{2}\sum_{k=1}^{n}[-D_{4k+1}(u) - D_{4k-2}(u) + D_{4k+2}(u) + D_{4k-1}(u) \\ &\qquad -D_{4k+3}(u) - D_{4k}(u) + D_{4k+4}(u) + D_{4k+1}(u)] \\ &= a[D_{4n+3}(u) - D_3(u)] \\ &+ \frac{1-2a}{2}[D_{4n+4}(u) - D_{4n+3}(u) + D_{4n+2}(u) - D_4(u) + D_3(u) - D_2(u)]. \end{aligned}$$

In process $n \to \infty$ we obtain

$$N_{a,2}(u) = -aD_3(u) + \frac{1-2a}{2}[-D_2(u) + D_3(u) - D_4(u)]. \tag{18}$$

For $N_{a,1}(u)$ in (14) we get by (17) the representation

$$\begin{aligned} N_{a,1}(u) &= a[D_0(u) + 2D_1(u) + 2D_2(u) + D_3(u)] \\ &+ \frac{1-2a}{2}[D_{-1}(u) + D_2(u) + D_0(u) + D_3(u) + D_1(u) + D_4(u)], \end{aligned}$$

which by (14) and (18) gives for $N_a(u)$ the following expression

$$\begin{aligned} N_a(u) &= a[D_0(u) + 2D_1(u) + 2D_2(u)] + \frac{1-2a}{2}[D_0(u) + 2D_1(u) + 2D_3(u)] \\ &= 2a[D_2(u) - D_3(u)] + \frac{1}{2}D_0(u) + D_1(u) + D_3(u). \end{aligned}$$

Using (16), we have for $0 \leqslant a \leqslant 3/8$

$$\begin{aligned} N_a(u) &= 2a[r_1(u-\frac{1}{2}) - r_3(u-\frac{1}{2})] \\ &+ 2r_0(u-\frac{1}{2}) + r_1(u-\frac{1}{2}) + r_2(u-\frac{1}{2}) + r_3(u-\frac{1}{2}) \end{aligned} \tag{19}$$

For sampling operators $B_{W,a}$ by (12) and (13) we obtain

$$\|B_{W,a}\| = \sup_{0 \leqslant u \leqslant 1} N_a(u) = \sup_{|u| \leqslant 1/2} N_a(u+1/2).$$

Since by (19) $N_a(u+1/2)$ is even, we write

$$\|B_{W,a}\| = \sup_{0 \leqslant u \leqslant 1/2} N_a(u+1/2). \tag{20}$$

From this point we are able to consider the cases $a = 3/8$ and $a = 0$ only. First, let $a = 3/8$. Then by (19) the integral representation of r_j yields

$$N_{3/8}(u+1/2) = \int_0^1 \left[2\cos\frac{1}{2}\pi t + \frac{7}{4}\cos\frac{3}{2}\pi t + \cos\frac{5}{2}\pi t + \frac{1}{4}\cos\frac{7}{2}\pi t\right]\cos\pi u t\, dt. \tag{21}$$

Denote

$$\begin{aligned} B(t) &:= 2\cos\frac{1}{2}\pi t + \frac{7}{4}\cos\frac{3}{2}\pi t + \cos\frac{5}{2}\pi t + \frac{1}{4}\cos\frac{7}{2}\pi t \\ &= \frac{1}{2}(1+\cos\pi t)\cos\frac{\pi t}{2}(1 + 2\cos\pi t + 2\cos 2\pi t). \end{aligned} \tag{22}$$

Since computations using Mathematica show that $B(t) \geqslant 0$ on $[0,2/5] \cup [4/5,1]$ and $B(t) \leqslant 0$ on $[2/5,4/5]$ we write

$$\begin{aligned} N_{3/8}(1/2) - N_{3/8}(u+1/2) &= \left(\int_0^{2/5} + \int_{2/5}^{4/5} + \int_{4/5}^{1}\right) B(t)(1-\cos\pi t u)dt \\ &\geqslant \left(\int_0^{2/5} + \int_{2/5}^{4/5}\right) B(t)(1-\cos\pi t u)dt. \end{aligned}$$

On the one hand, an elementary inequality $\sin \pi x \leqslant \pi x$ for $x \geqslant 0$ gives for $0 \leqslant t \leqslant 1$, $0 \leqslant u \leqslant 1/2$, that

$$1 - \cos \pi t u = 2 \sin^2 \frac{\pi t u}{2} \leqslant \frac{\pi^2}{2} u^2 t^2. \tag{23}$$

On the other hand $\operatorname{sinc}(x)$ is decreasing on $[0, 1/2]$, therefore we obtain for $0 \leqslant t \leqslant 2/5,\ 0 \leqslant u \leqslant 1/2$

$$\begin{aligned} 1 - \cos \pi t u &= 2 \sin^2 \frac{\pi t u}{2} = \frac{\pi^2 u^2 t^2}{2} \operatorname{sinc}^2 \frac{t u}{2} \\ &\geqslant \frac{\pi^2 u^2 t^2}{2} \operatorname{sinc}^2 \frac{1}{10} = \frac{75 - 25\sqrt{5}}{4} u^2 t^2. \end{aligned} \tag{24}$$

Using computations with Mathematica these estimates yield

$$\begin{aligned} N_{3/8}(1/2) - N_{3/8}(u + 1/2) &\geqslant \frac{75 - 25\sqrt{5}}{4} u^2 \int_0^{2/5} t^2 B(t) dt + \frac{\pi^2}{2} u^2 \int_{2/5}^{4/5} t^2 B(t) dt \\ &= \frac{u^2}{2} (0.251... - 0.223...) \geqslant 0. \end{aligned}$$

Now the proof for the case $a = 3/8$ is completed by (20), (19) and (2), that is

$$\|B_{W,3/8}\| = \sup_{0 \leqslant u \leqslant 1/2} N_{3/8}(u + 1/2) = N_{3/8}(1/2) = \frac{332}{105\pi}.$$

The same scheme of the proof gives the result in the case $a = 0$. ∎

To consider the norm $\|B_{W,a}\|$ for all values $a \in [0, 3/8]$ we deduce a simple but effective upper bound for $\|B_{W,a}\|$.

Theorem 2. *For any three parameters $a_0, a_1, a \in \mathbb{R}$ with $a_0 \neq a_1$ we have*

$$\|B_{W,a}\| \leqslant \left| \frac{a_1 - a}{a_1 - a_0} \right| \|B_{W,a_0}\| + \left| \frac{a - a_0}{a_1 - a_0} \right| \|B_{W,a_1}\|.$$

In particular, if $0 = a_0 \leqslant a \leqslant a_1 = 3/8$, then

$$\|B_{W,a}\| \leqslant \frac{362 - 80a}{105\pi}.$$

Proof. It is easy to check that for any $a_0, a_1, a \in \mathbb{R}$ with $a_0 \neq a_1$ the Blackman window $\lambda_{B,a}$ enjoys the interpolation formula

$$\lambda_{B,a}(t) = \frac{a_1 - a}{a_1 - a_0} \lambda_{B,a_0}(t) + \frac{a - a_0}{a_1 - a_0} \lambda_{B,a_1}(t). \tag{25}$$

From this by (9) and (1) we obtain

$$B_{W,a} f = \frac{a_1 - a}{a_1 - a_0} B_{W,a_0} f + \frac{a - a_0}{a_1 - a_0} B_{W,a_1} f \tag{26}$$

for any $f \in C(\mathbb{R})$ which gives our assertions. ■

In fact we can show that the upper bound in Theorem 2 is sharp for any $a \in [0,3/8]$.

Theorem 3. *For* $0 \leqslant a \leqslant 3/8$

$$\|B_{W,a}\| = \frac{362-80a}{105\pi}.$$

Proof. By Theorem 2 we have to show for $0 \leqslant a \leqslant 3/8$ that

$$\|B_{W,a}\| \geqslant \frac{362-80a}{105\pi}.$$

From (12) and (13) we get

$$\|B_{W,a}\| \geqslant \sum_{k=-\infty}^{\infty} |s_{B,a}(k-\frac{1}{2})| = N_a(1/2).$$

In the case $0 \leqslant a \leqslant 3/8$ by (19) and (4) we obtain

$$N_a(1/2) = 2a[r_1(0)-r_3(0)] + 2r_0(0) + r_1(0) + r_2(0) + r_3(0) = \frac{362-80a}{105\pi}. \blacksquare$$

By the representation (10) we see that the window $\lambda_{B,27/64} = \lambda_{B,0.4218...}$, close to the classical Blackman window $\lambda_{B,0.42}$, might be an another easy case to compute the norm $\|B_{W,a}\|$. For the proof we used the scheme described before Theorem 1.

Theorem 4. *We have*

$$\|B_{W,27/64}\| = \|B_{W,0.4218...}\| = \frac{3973}{1260\pi} = 1.0036...$$

3. ORDER OF APPROXIMATIONS BY BLACKMAN-TYPE SAMPLING SERIES

In this section we will find the order of approximation by Blackman-type sampling series.

Theorem 5. *If* $B_{W,a}$ *is the sampling operator for* $f \in C(\mathbb{R})$ *defined by* (1), *then for some* $M_a > 0$

$$\|f - B_{W,a}f\|_C \leqslant M_a \omega_2(f, \frac{1}{W})$$

uniformly in $W \geqslant 4/\pi$.

Proof. Let $g \in B^{\infty}_{\sigma}$ $(\sigma < \pi W)$. Then by (5) we have $S^{\mathrm{sinc}}_W g = g$. Motivated by (11) we denote

$$f_{W,a}(t) := af(t) + \frac{1}{4}[f(t-\frac{1}{W}) + f(t+\frac{1}{W})] + \frac{1-2a}{4}[f(t-\frac{2}{W}) + f(t+\frac{2}{W})]$$

and then

$$\begin{aligned} f - B_{W,a}f &= f - B_{W,a}(f-g+g) = f - B_{W,a}(f-g) - g_{W,a} \\ &= f - f_{W,a} - B_{W,a}(f-g) - (g_{W,a} - f_{W,a}). \end{aligned} \tag{27}$$

We get

$$\begin{aligned} & f(t)-f_{W,a}(t) \\ = \; & f(t)-af(t)-\frac{1}{4}[f(t-\frac{1}{W})+f(t+\frac{1}{W})]-\frac{1-2a}{4}[f(t-\frac{2}{W})+f(t+\frac{2}{W})] \\ = \; & -\frac{1}{4}[f(t-\frac{1}{W})-2f(t)+f(t+\frac{1}{W})]-\frac{1-2a}{4}[f(t-\frac{2}{W})-2f(t)+f(t+\frac{2}{W})] \\ = \; & -\frac{1}{4}\Delta^2_{1/W}f(t-\frac{1}{W})-\frac{1-2a}{4}\Delta^2_{2/W}f(t-\frac{2}{W}). \end{aligned} \tag{28}$$

By the definition of the modulus of continuity we have

$$\|\Delta^m_h f\|_C \leqslant \omega_m(f,h)$$

and therefore by (27), (28) (recall that $h=1/W$) and (7)

$$\begin{aligned} \|f-B_{W,a}f\|_C &\leqslant \|f-f_{W,a}\|_C+\|B_{W,a}\|\|f-g\|_C+\|f_{W,a}-g_{W,a}\|_C \\ &\leqslant \frac{1}{4}\omega_2\left(f,\frac{1}{W}\right)+\frac{|1-2a|}{4}\omega_2\left(f,\frac{2}{W}\right) \\ &+ \left(\|B_{W,a}\|+|a|+\frac{1}{2}+\frac{|1-2a|}{2}+\right)\|f-g\|_C \\ &\leqslant (C_{1,a}+\|B_{W,a}\|)\|f-g\|_C+C_{2,a}\omega_2\left(f,\frac{1}{W}\right). \end{aligned} \tag{29}$$

Now let us take in (29) the function $g=g^*\in B^\infty_\sigma$ ($2\leqslant\sigma=\varepsilon\pi W<\pi W$, $0<\varepsilon<1$) as in Proposition 1. We have by (7)

$$\|f-g^*\|\leqslant C_2\omega_2\left(f,\frac{1}{\varepsilon\pi W}\right)\leqslant C_2\left(1+\frac{1}{\varepsilon\pi}\right)^2\omega_2\left(f,\frac{1}{W}\right),$$

and the proof is completed due to (29). ■

Since in the case $a=5/8$ from (28) we get

$$\begin{aligned} f(t)-f_{W,5/8}(t)&=\frac{1}{16}[f(t-\frac{2}{W})+f(t+\frac{2}{W})]-\frac{1}{4}[f(t-\frac{1}{W})+f(t+\frac{1}{W})]+\frac{3}{4}f(t) \\ &=\frac{1}{16}\Delta^4_{1/W}f(t-\frac{2}{W}), \end{aligned}$$

then following the scheme of the proof of Theorem 5 we can achieve much better order of approximation.

Theorem 6. *The Blackman sampling operator $B_{W,5/8}$, defined by* (1) *enjoys the inequality*

$$\|f-B_{W,5/8}f\|_C\leqslant M\omega_4(f,\frac{1}{W})$$

uniformly in $W\geqslant 4/\pi$.

As we see, the Blackman sampling operator $B_{W,5/8}$ gives a good order of approximation. Let us estimate the norm of that sampling operator.

In general for $\|B_{W,a}\|$ more tight lower bound then in the proof of Theorem 3 we get from the following estimation.

Theorem 7. *For all* $a \in \mathbb{R}$

$$\|B_{W,a}\| \geqslant 2\sum_{k=1}^{4} |s_{B,a}(1/2-k)| + 2\left|s_{B,a}(7/2) + \frac{1}{9\pi} - \frac{88a}{315\pi}\right|.$$

Proof. By (12) we have

$$\begin{aligned}\|B_{W,a}\| &\geqslant \sum_{k=-\infty}^{\infty} |s_{B,a}(1/2-k)| = 2\sum_{k=1}^{\infty} |s_{B,a}(1/2-k)| \\ &\geqslant 2\sum_{k=1}^{4} |s_{B,a}(1/2-k)| + 2|\sum_{k=5}^{\infty} (-1)^{k+1} s_{B,a}(1/2-k)|.\end{aligned}$$

The definition (14) of $N_{a,2}$ gives

$$N_{a,2}(1/2) = 2\sum_{k=5}^{\infty} (-1)^{k+1} s_{B,a}(1/2-k) - 2s_{B,a}(7/2).$$

Therefore,

$$\|B_{W,a}\| \geqslant 2\sum_{k=1}^{4} |s_{B,a}(1/2-k)| + |N_{a,2}(1/2) + 2s_{B,a}(7/2)|.$$

Since by (16), (18) and (4)

$$N_{a,2}(1/2) = \frac{2}{\pi}\left(\frac{1}{9} - \frac{88a}{315}\right),$$

we obtain our assertion. ∎

Corollary. *The Blackman sampling operator* $B_{W,5/8}$*, defined by* (1) *enjoys the inequality*

$$1.18\ldots = \frac{56}{15\pi} \leqslant \|B_{W,5/8}\| \leqslant \frac{176}{35\pi} = 1.60\ldots$$

Proof. The lower bound for $\|B_{W,5/8}\|$ we compute by Theorem 7. For the upper bound we use Theorem 2. The computations with Mathematica show, that the minimal value results when we take $a_0 = 0$ and $a_1 = 1/2$ in Theorem 2. ∎

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A Construction of Wavelets

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Abstract

In this note, given a multiresolution analysis, we construct a class of four-coefficient scaling filters using only elementary algebraic operations. The method of construction also reveals that the sum conditions $\sum a_{even} = \sum a_{odd} = 1$ can also be verified without referring to the vanishing moment property.

1 A Review of Construction of Wavelets

We begin this note by recalling the definition of a multiresolution analysis.

Definition A *multiresolution analysis* $\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$ with scaling function φ is an increasing sequence of subspaces of $L^2(R)$ satisfying the following four conditions:

1. (density) $\cup_j V_j$ is dense in $L^2(R)$,
2. (separation) $\cap_j V_j = \{0\}$,
3. (scaling) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$,
4. (orthonormality) $\{\varphi(x-\gamma)\}_{\gamma \in Z}$ is an orthonormal basis for V_0.

First of all, $\{2^{j/2}\varphi(2^j x-\gamma)\}_{\gamma\in Z}$ forms an orthonormal basis for V_j. This is evident from the definition. In order to form an orthonormal basis for $L^2(R)$, the density condition 1 seems to suggest, at first, to combine all the orthonormal bases $\{2^{j/2}\varphi(2^j x-\gamma)\}_{\gamma\in Z}$ of V_j. But this does not work since there are distinct elements from two orthonormal bases, $\{2^{j/2}\varphi(2^j x-\gamma)\}_{\gamma\in Z}$ for V_j and $\{2^{(j+1)/2}\varphi(2^{j+1}x-\gamma)\}_{\gamma\in Z}$ for V_{j+1}, that are not orthogonal to each other. What is required at this point is the construction of an orthonormal basis for the orthogonal complement W_0 of V_0 in V_1. More generally, we need to find an orthonormal basis for W_j where

$$V_{j+1} = V_j \oplus W_j,$$

for $j = 0, 1, \ldots$ The function ψ for which $\{\psi(x-\gamma)\}_{\gamma\in Z}$ is an orthonormal basis for W_0 is the *wavelet generator* . Once ψ is found, then $\{2^{j/2}\psi(2^j x-\gamma)\}_{\gamma\in Z}$ form an orthonormal basis for W_j and as

$$L^2(R) = V_0 \oplus (\oplus_{j=0}^{\infty} W_j) = \oplus_{j\in Z} W_j,$$

$\{\varphi(x-\gamma)\}_{\gamma\in Z} \cup \{2^{j/2}\psi(2^j x-\gamma)\}_{\gamma\in Z, j\geq 0}$ or $\{2^{j/2}\psi(2^j x-\gamma)\}_{\gamma\in Z, j\geq 0}$ form an orthonormal wavelet basis for $L^2(R)$.

Since $\varphi \in V_0 \subseteq V_1$, we must have the following *scaling identity* ,

$$\varphi(x) = \sum_{\gamma\in Z} a_\gamma \varphi(2x-\gamma). \tag{1.1}$$

It is well known [1] that the conditions that must be met by the coefficients a_i's are the following:

$$\sum_{\gamma\in Z} |a_\gamma|^2 = 2, \tag{1.2}$$

$$\sum_{\gamma'\in Z} a_{\gamma'} a_{2\gamma+\gamma'} = 2\delta(\gamma, 0), \tag{1.3}$$

and

$$\sum_{\gamma\in Z} a_\gamma = 2. \tag{1.4}$$

Equation (1.2) is a consequence of the scaling identity and of the fact that $\|\varphi\|_2 = 1$. Equation (1.3) is obtained from condition 4 of the definition, -i.e.,

$$\int \varphi(x-\gamma)\varphi(x)\, dx = \delta(\gamma, 0),$$

upon substituting the scaling identities for $\varphi(x-\gamma)$ and $\varphi(x)$. Equation (1.4) is obtained also from the scaling identity with additional condition that $\int \varphi(x)dx \neq 0$. Namely, integrate both

sides of (1.1) and make a change of variable. Note that (1.2) is a special case of (1.3). Clearly, the scaling function determines a multiresolution analysis, but not conversely. A construction of wavelets involves reversing the procedure. In other words, we need a characterization of a function that satisfies the scaling identity (1.1) and that generates a multiresolution analysis. In order to better explain the content of the present note which is described in the next section, let us review below in five steps how a general construction of wavelets is done. We refer the reader to [1] and [3] for a more complete descriptions of these steps.

Step 1: This step consists of producing solutions to algebraic identities (1.2)-(1.4).

Step 2: Once a_i's are determined, a possible scaling function φ is defined. This can be done, for instance, by finding a fixed point of the linear transformation

$$Sf(x) = \sum_{\gamma \in Z} a_\gamma f(2x - \gamma)$$

by iterations

$$\varphi = \lim_{n \to \infty} S^n f$$

with an appropriate initial function f.

Step 3: Now, we must verify that the function φ, defined in Step 2, generates a multiresolution analysis. To this end, we let

$$A_0(\xi) = \frac{1}{2} \sum_{\gamma \in Z} a_\gamma e^{2\pi i \gamma \xi}.$$

It turns out that the following condition (1.5) along with (1.2)-(1.4) serve as sufficient conditions for the orthonomality of $\{\varphi(x - \gamma)\}_{\gamma \in Z}$.

$$A_0(\xi) \neq 0 \qquad for \ \ |\xi| \leq \frac{1}{4}. \tag{1.5}$$

Step 4: This step is to study those conditions that are necessary to construct wavelets. Let $\psi_0 = \varphi$ and $\psi_1 = \psi$, a wavelet generator to be determined. Also, let $a_\gamma^0 = a_\gamma$, the solutions of (1.2)-(1.4). Since $\{\psi_k(x - \gamma)\}_{\gamma \in Z, k=0,1}$ must be an orthonormal basis for $V_1 = V_0 \oplus W_0$, noting that $\{\varphi(2x - \gamma)\}_{\gamma \in Z}$ is an orthonormal basis for V_1,

$$\psi_k(x) = \sum_{\gamma \in Z} a_\gamma^k \varphi(2x - \gamma), \qquad k = 0, 1. \tag{1.6}$$

Equation (1.6) with $k = 0$ is the scaling identity (1.1). The condition that $\psi_0 \perp \psi_1$, or more precisely, with $j, k \in \{0, 1\}$, $\gamma \in Z$,

$$\int \psi_j(x - \gamma) \psi_k(x) dx = \delta(j, k) \delta(\gamma, 0),$$

leads to, upon replacing ψ_0 and ψ_1 by the corresponding expressions in (1.6),

$$\sum_{\gamma' \in Z} a^j_{\gamma'} a^k_{2\gamma+\gamma'} = 2\delta(j,k)\delta(\gamma,0). \tag{1.7}$$

Equation (1.7) contains (1.2) and (1.3) as special cases. We can not expect the wavelet generator ψ to satisfy $\int \psi(x)dx \neq 0$, so we do not have an equivalent condition to (1.4) for a^1_γ. Hence, we can only use (1.4), restated below in the current notation as

$$\sum_{\gamma \in Z} a^0_\gamma = 2. \tag{1.8}$$

At this point, we have reduced the problem of constructing wavelets to the solution of algebraic identities (1.7) and (1.8), together with condition (1.5), which was to guarantee the orthonomality condition 4 for φ. In summary, we have (Theorem 4.3, [3]) that

Theorem 1.1 *Suppose φ generates a multiresolution analysis and a^k_γ satisfy (1.7) and (1.8) with ψ_k defined by (1.6) and $\psi_0 = \varphi$. Then the functions $\{2^{j/2}\psi_1(2^j x - \gamma)\}$ for $j \in Z$, $\gamma \in Z$ form an orthonormal basis of $L^2(R)$.*

Step 5: This step completes the task that began in Step 4. Namely, we need to produce the solutions of (1.7) and (1.8) that satisfy (1.5). The function A_0 defined earlier can be written as

$$A_0(\xi) = \frac{1}{2}\sum_{\gamma \in Z} a^0_\gamma e^{2\pi i \gamma \xi}.$$

Similarly, we define

$$A_1(\xi) = \frac{1}{2}\sum_{\gamma \in Z} a^1_\gamma e^{2\pi i \gamma \xi}.$$

In terms of A_0 and A_1, equations (1.7) and (1.8) can be shown to be equivalent to

$$\sum_{p=1}^{2} A_k(\xi+\eta_p)A_j(\xi+\eta_p) = \delta(j,k), \tag{1.9}$$

and

$$A_0(0) = 1, \tag{1.10}$$

respectively [1] where $\eta_1 = 0$ and $\eta_2 = \frac{1}{2}$. One method of construction presented by Daubechies is to first solve for $\{a^0_\gamma\}$ using

$$A_0(0) = 1$$

$$|A_0(\xi)|^2 + |A_0(\xi + \frac{1}{2})|^2 = 1$$

$$A_0(\xi) \neq 0 \qquad \text{for } |\xi| \leq \tfrac{1}{4},$$

and subsequently incorporate the remaining equations from (1.9) to find $A_1(\xi)$ which in turn gives $\{a_\gamma^1\}$. The remaining equations are

$$|A_1(\xi)|^2 + |A_1(\xi + \frac{1}{2})|^2 = 1, \tag{1.11}$$

and

$$A_0(\xi)A_1(\xi) + A_0(\xi + \frac{1}{2})A_1(\xi + \frac{1}{2}) = 0. \tag{1.12}$$

It is shown [1] that

$$A_1(\xi) = e^{2\pi i\xi} A_0(\xi + \frac{1}{2})$$

solves (1.11) and (1.12) and this amounts to setting

$$a_\gamma^1 = (-1)^{\gamma+1} a_{1-\gamma}^0. \tag{1.13}$$

2 A Construction of Scaling Functions and Their Wavelets

In this section, we present a method of constructing a class of four-coefficient scaling filters using only elementary algebraic operations. Particularly, we are interested in obtaining the solutions $a_\gamma \equiv a_\gamma^0$ of equations (1.2), (1.3) and (1.4). Recall from the previous section, each solution will generate a multiresolution analysis provided that condition (1.5) holds. Formulas for constructing four-coefficient scaling filters already exist. An elegant approach of Daubechie is well documented in [1]. Therefore, we do not claim that the results obtained in this section are new. But rather, the purpose of the present note is to shed another perspective in constructions of scaling filters and to demonstrate the fact that the sum condition $\sum a_{even} = \sum a_{odd} = 1$ can be derived without referring to the vanishing moment property. The fact that the sum condition is a consequence of (2.1), which will be established in this section, appears to be new. We do not discuss a complete construction of wavelets in this note. However, once $a_\gamma \equiv a_\gamma^0$ are found, the associated wavelets can be generated simply from (1.13).

We consider the case where only a_0, a_1, a_2 and a_3 are the terms which could possibly be

nonzero. We also assume that they are real. Equations (1.2), (1.4) and (1.3) become respectively,

$$\begin{aligned} a_0^2 + a_1^2 + a_2^2 + a_3^2 &= 2 \\ a_0 + a_1 + a_2 + a_3 &= 2 \\ a_0 a_2 + a_1 a_3 &= 0. \end{aligned} \tag{2.1}$$

If $a_3 \neq 0$, then $a_1 = -\frac{a_0 a_2}{a_3}$, and substituting, we get

$$\begin{aligned} a_0^2 + \tfrac{a_0^2 a_2^2}{a_3^2} + a_2^2 + a_3^2 &= 2 \\ a_0 - \tfrac{a_0 a_2}{a_3} + a_2 + a_3 &= 2. \end{aligned}$$

Let $x \equiv a_0$, $y \equiv a_2$ and $c \equiv \frac{1}{a_3}$, thereby obtaining

$$\begin{aligned} x^2 + c^2 x^2 y^2 + y^2 &= 2 - \tfrac{1}{c^2} \\ x - cxy + y &= 2 - \tfrac{1}{c}. \end{aligned} \tag{2.2}$$

Assuming $1 - cx \neq 0$, (2.2) yields

$$y = \frac{2 - \frac{1}{c} - x}{1 - cx} \quad \text{and} \quad y^2 = \frac{2 - \frac{1}{c^2} - x^2}{1 + c^2 x^2}. \tag{2.3}$$

Squaring the first equation and equating it to the second equation yields

$$2c^2x^4 - 4c^2x^3 + (2c^2 - 4c + 4)x^2 + 4(c-1)x + 2 - \frac{4}{c} + \frac{2}{c^2} = 0. \tag{2.4}$$

In order to study the solutions of (2.4), let

$$f(x) = 2c^2x^4 - 4c^2x^3 + (2c^2 - 4c + 4)x^2 + 4(c-1)x + 2 - \frac{4}{c} + \frac{2}{c^2}.$$

First, we observe that, for every $c \neq 0$,

$$f'(\frac{1}{2}) = 0, \tag{2.5}$$

and that

$$f(x + \frac{1}{2}) = 2c^2x^4 + (4 - 4c - c^2)x^2 + \frac{1}{8}c^2 + c + 1 - \frac{4}{c} + \frac{2}{c^2}. \tag{2.6}$$

Equation (2.6) shows that the graph of f is symmetric about the line $x = \frac{1}{2}$. Let $\tilde{f}(x) \equiv f(x + \frac{1}{2})$. Then $\tilde{f}'(x) = 0$ has roots

$$0 \quad \text{and} \quad \pm \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}.$$

When $\pm\frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ are real, -i.e., for

$$c < -2 - 2\sqrt{2} \quad \text{or} \quad c > -2 + 2\sqrt{2}, \tag{2.7}$$

$\tilde{f}$ attains its minimum at $\pm\frac{1}{2}\sqrt{1+\frac{4}{c}-\frac{4}{c^2}}$. For the values of c specified in (2.7), we have $\tilde{f}(\pm\frac{1}{2}\sqrt{1+\frac{4}{c}-\frac{4}{c^2}})=0$. Translating back into f, we conclude that

$$\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{4}{c}-\frac{4}{c^2}}, \tag{2.8}$$

are the roots of $f(x)=0$. Each is a double root of the quartic $f(x)=0$. If we denote two roots in (2.8) by x_1 and x_2, then notice that $x_2=1-x_1$. This, of course, is a rehash of the fact that the graph of f is symmetric about $x=\frac{1}{2}$. The following theorem serves as the first step toward establishing the algorithm that generates many wavelets.

Theorem 2.1 *Consider the equations in (2.1) and let* $x \equiv a_0$, $y \equiv a_2$ *and* $c \equiv \frac{1}{a_3}$. *Then* $x + y = 1$.

Proof: Since $x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ and $x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4}{c} - \frac{4}{c^2}}$ are two roots of $f(x) = 0$,

$$(x - \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{c^2} + \frac{4}{c}})(x - \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{c^2} + \frac{4}{c}}),$$

are two linear factors of $f(x)$, and upon multiplying and simplifying, we obtain $x^2 - x + \frac{1}{c^2} - \frac{1}{c}$. If $x = x_1$ or $x = x_2$, then $x^2 - x + \frac{1}{c^2} - \frac{1}{c} = 0$, which reduces

$$cx - 1 = cx^2 + \frac{1}{c} - 2. \tag{2.9}$$

But we have, from (2.3) and (2.9),

$$x + y = x + \frac{2 - \frac{1}{c} - x}{1 - cx} = \frac{-cx^2 - \frac{1}{c} + 2}{1 - cx} = 1.$$

□

Note that from (2.1) and Theorem 2.1, we obtain

$$a_0 + a_2 = 1 \quad \text{and} \quad a_1 + a_3 = 1. \tag{2.10}$$

Note also that (1.13) implies $A_0(\frac{1}{2}) = 0$ and $A_1(0) = 0$, which is the well known vanishing moment condition for wavelets. $A_0(\frac{1}{2}) = 0$ is satisfied by (2.10). And this is what is done generally for construction of wavelets. More specifically, (2.1) and (1.13), and hence (2.10), are solved together to get the complete descriptions of scaling filters as well as wavelets. As pointed out earlier, the task of this note has been different. We examined the solutions of (2.1) independently, and derived (2.10) without referring to (1.13). In [2] (p. 296), it is noted that "the sum condition $\sum a_{even} = \sum a_{odd} = 1$ is always imposed". Theorem 2.1 tells us that, with wavelets whose scaling relation involves four terms, it is necessary that $\sum a_{even} = \sum a_{odd} = 1$. Some other immediate consequences of Theorem 2.1 are:

$$\begin{aligned}
(a_0 - \tfrac{1}{2})^2 + (a_1 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_2 - \tfrac{1}{2})^2 + (a_2 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_0 - \tfrac{1}{2})^2 + (a_3 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
(a_2 - \tfrac{1}{2})^2 + (a_3 - \tfrac{1}{2})^2 &= \tfrac{1}{2} \\
a_1^2 - a_1 - a_0 a_2 &= 0 \\
a_3^2 - a_3 - a_0 a_2 &= 0.
\end{aligned}$$

Now, we are ready to present the algorithm for the solution of our problem:

ALGORITHM:

1. Choose $a_3 \in (\frac{1}{-2-2\sqrt{2}}, \frac{1}{-2+2\sqrt{2}})$.
2. Find $a_0 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_3 - 4a_3^2}$.
3. Get $a_2 = 1 - a_0$.
4. Get $a_1 = 1 - a_3$.

The corresponding scaling function φ can be derived by finding a fixed point of the mapping

$$Sf(x) = a_0 f(2x) + a_1 f(2x-1) + a_2 f(2x-2) + a_3 f(2x-3),$$

by iteration with a reasonable starting function, f^0, -i.e.,

$$\varphi(x) = \lim_{n\to\infty} S^n f^0(x).$$

To install p accuracy condition on wavelets, -i.e., to establish the p vanishing moment properties of wavelets, it is required [2] that

$$\sum (-1)^k k^m a_k = 0, \qquad \text{for } m < p. \tag{2.11}$$

By (2.10), (2.11) is satisfied with $m = 0$. Therefore, the algorithm, when a_3 is taken in the range specified, will generates a family of wavelets whose accuracy order is 1. If we want the second order accuracy in wavelets, then we must also have equation (2.11) with $m = 1$. In our case, this is

$$-a_1 + 2a_2 - 3a_3 = 0. \tag{2.12}$$

Since $a_1 = 1 - a_3$ and $a_0 = 1 - a_2$, substituting into (2.12), we obtain $2a_0 = 1 - 2a_3$. Replacing a_0 by $\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_3 - 4a_3^2}$ and solving for a_3, we obtain $a_3 = \frac{1\pm\sqrt{3}}{4}$. If we take $a_3 = \frac{1-\sqrt{3}}{4}$, then the following wavelet is generated by the algorithm;

$$a_0 = \frac{1+\sqrt{3}}{4}, \quad a_1 = \frac{3+\sqrt{3}}{4}, \quad a_2 = \frac{3-\sqrt{3}}{4}, \quad a_3 = \frac{1-\sqrt{3}}{4}.$$

Of course, this is one of Daubechies wavelets. It is important to note that our approach can be extended to any scaling function that involves four nonzero coefficients. For instance, consider the case;

$$\varphi(x) = a_{-1}\varphi(2x+1) + a_0\varphi(2x) + a_1\varphi(2x-1) + a_2\varphi(2x-2).$$

Then the algorithm is modified to

1. Choose $a_2 \in (\frac{1}{-2-2\sqrt{2}}, \frac{1}{-2+2\sqrt{2}})$.
2. Find $a_{-1} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4a_2 - 4a_2^2}$.
3. Get $a_1 = 1 - a_{-1}$.
4. Get $a_0 = 1 - a_2$.

Arguing as before to obtain the second order accuracy in the wavelet, we get once again the coefficients,

$$a_{-1} = \frac{1+\sqrt{3}}{4}, \quad a_0 = \frac{3+\sqrt{3}}{4}, \quad a_1 = \frac{3-\sqrt{3}}{4}, \quad a_2 = \frac{1-\sqrt{3}}{4}.$$

The support of the scaling function is $[-1, 2]$. Also, we note that choosing $a_2 = 0$ and the minus sign for a_{-1} in the algorithm, yields $a_{-1} = a_2 = 0$, $a_0 = a_1 = 1$. These values yield the scaling function φ for the classical Haar wavelets.

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On $C^{1,1}$ optimization problems

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Abstract

Many definitions of second order generalized derivatives have been introduced to study optimization problems involving $C^{1,1}$ data. The aim of this paper is to show some relations among these definitions and to prove necessary and sufficient conditions for $C^{1,1}$ optimization problems.

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1 Introduction

The class of $C^{1,1}$ functions, that is the class of differentiable functions with a locally Lipschitzian gradient, was first studied by Hiriart-Urruty, Strodiot and Hien Nguyen in [19]. Several problems of applied mathematics (variational inequalities, semi-infinite programming, penalty functions, augmented lagrangian, proximal point methods, iterated local minimization by decomposition) involve differentiable functions with no hope of being twice differentiable but with locally Lipschitzian gradient; for this class of functions many second order generalized derivatives have been introduced to obtain generalized optimality conditions (see the references). In this paper we will give some relations among several definitions that one can find in literature and we will study necessary and sufficient conditions for optimization problems expressed by means of these derivatives.

2 Preliminary definitions and results

In the following Ω will denote an open subset of $\mathbb{R}^n$.

Definition 2.1. *A function $f : \Omega \to \mathbb{R}$ is of class $C^{1,1}$, or briefly a $C^{1,1}$ function, at x_0 if it is differentiable and $\nabla f(x)$ is locally Lipschitzian at x_0, i.e. there exists a constant $K > 0$ such that:*

$$\|\nabla f(x) - \nabla f(y)\| \leq K\|x - y\|,$$

for all x and y in a neighborhood U of x_0.

Example 2.1. *Let $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^2 on Ω and consider $f(x) = [g^+(x)]^2$ where $g^+(x) = \max\{g(x), 0\}$. This type of functions arises in some penalty methods. It is easy to prove that f is a $C^{1,1}$ function on Ω [19].*

We now remember the notions of second order generalized derivative on which we will focus our attention.

Definition 2.2. *Let us consider a function $f : \Omega \to \mathbb{R}$ of class $C^{1,1}$ at x_0.*

i) Peano's second upper derivative of f at x_0 in the direction $d \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_P(x_0; d) = 2 \limsup_{t \downarrow 0} \frac{f(x_0 + td) - f(x_0) - t\nabla f(x_0)d}{t^2}.$$

ii) Dini-Hadamard's second upper derivative of f at x_0 in the direction $d \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_H(x_0; d) = 2 \limsup_{t \downarrow 0, d' \to d} \frac{f(x_0 + td') - f(x_0) - t\nabla f(x_0)d'}{t^2}.$$

iii) Riemann's second upper derivative of f at x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_R(x_0; d, w) = \limsup_{t \downarrow 0} \frac{f(x_0 + td + tw) - f(x_0 + td) - f(x_0 + tw) + f(x_0)}{t^2}.$$

iv) Dini's second upper derivative of f at x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_D(x_0; d, w) = \limsup_{t \downarrow 0} \frac{\nabla f(x + td)w - \nabla f(x)w}{t}.$$

v) Yang-Jeyakumar's second upper derivative of f at x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_Y(x_0; u, v) = \sup_{z \in \mathbb{R}^n} \limsup_{t \downarrow 0} \frac{\nabla f(x_0 + td + tz)w - \nabla f(x_0 + tz)w}{t}.$$

vi) Clarke's second upper derivative of f at x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_C(x_0; d, w) = \limsup_{x' \to x,\ t \downarrow 0} \frac{\nabla f(x' + td)w - \nabla f(x')w}{t}.$$

vii) BenTal-Zowe's second upper derivative of f at x_0 in the directions $d, w \in \mathbb{R}^n$ is defined as:

$$\overline{f}''_B(x_0; d, w) = 2 \limsup_{t \downarrow 0} \frac{f(x_0 + td + 2^{-1}t^2 w) - f(x_0) - t\nabla f(x_0)d}{t^2}.$$

Remark 2.1. *In an analogous way one can define lower derivatives and we denote them by $\underline{f}''$.*

In the following results we will give some relations among the derivatives of definition 2.2.

Remark 2.2. *In [49] is given the following chain of inequalities:*

$$\overline{f}''_D(x_0; d, w) \leq \overline{f}''_Y(x_0; d, w) \leq \overline{f}''_C(x_0; d, w).$$

Furthermore $\overline{f}''_Y(x_0; d, w) = \overline{f}''_C(x_0; d, w)$ if and only if the map $\overline{f}''_Y(\cdot; d, w)$ is upper semicontinuous .

Remark 2.3. *In [9] is given the following characterization of Clarke's generalized derivative:*

$$\overline{f}''_C(x_0; d, w) = \limsup_{x \to x_0,\ s,t \downarrow 0} \frac{\overline{\Delta}_2^{d,w} f(x; s, t)}{st}$$

where:

$$\overline{\Delta}_2^{d,w} f(x; s, t) = f(x + sd + tw) - f(x + sd) - f(x + tw) + f(x).$$

From this characterization one can trivially deduce that $\overline{f}''_R(x_0; d, w) \leq \overline{f}''_C(x_0; d, w)$.

Proposition 2.1. *Let f be a function of class $C^{1,1}$ at x_0. Then:*

$$\overline{f}''_B(x_0; d, w) = \nabla f(x_0)w + \overline{f}''_P(x_0; d).$$

Proof. Let $l_B \in \partial_B^2 f(x_0; d, w)$ and $l_P \in \partial_P^2 f(x_0; d)$ where:

$$\partial_B^2 f(x_0; d, w) = \left\{ l = 2 \lim_{k \to +\infty} \frac{f(x_0 + t_k d + 2^{-1}t_k^2 w) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2}, t_k \downarrow 0 \right\}$$

$$\partial_P^2 f(x_0; d) = \left\{ l = 2 \lim_{k \to +\infty} \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2}, t_k \downarrow 0 \right\}$$

Eventually by extracting subsequences, we have:

$$l_B = 2 \lim_{k \to +\infty} \frac{f(x_0 + t_k d + 2^{-1}t_k^2 w) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2} =$$

$$2 \lim_{k \to +\infty} \frac{f(x_0 + t_k d + 2^{-1}t_k^2 w) - f(x_0 + t_k d)}{t_k^2} + \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2} =$$

$$2 \lim_{k \to +\infty} \nabla f(\xi_k)w + \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2} = \nabla f(x_0)w + l_P$$

□

Lemma 2.1. *Let f be a function of class $C^{1,1}$ at x_0. Then $\overline{f}''_P(x_0;d) = \overline{f}''_H(x_0;d)$ and $\underline{f}''_P(x_0;d) = \underline{f}''_H(x_0;d)$.*

Proof. It is trivial that $\overline{f}''_P(x_0;d) \leq \overline{f}''_H(x_0;d)$. Vice versa, let $l \in \partial^2_H f(x_0;d)$ where:

$$\partial^2_H f(x_0;d) := \{l = \lim_{k\to+\infty} 2\frac{f(x_0+t_kd_k)-f(x_0)-t_k\nabla f(x_0)d_k}{t_k^2}, t_k \downarrow 0, d_k \to d\}.$$

Then there exist $t_k \downarrow 0$, $d_k \to d$ s.t.:

$$\frac{l}{2} = \lim_{k\to+\infty} \frac{f(x_0+t_kd_k)-f(x_0)-t_k\nabla f(x_0)d_k}{t_k^2} =$$

$$\lim_{k\to+\infty} \frac{f(x_0+t_kd_k)-f(x_0+t_kd)+f(x_0+t_kd)-f(x_0)-t_k\nabla f(x_0)d-t_k\nabla f(x_0)(d_k-d)}{t_k^2}.$$

Taking eventually a subsequence:

$$\lim_{k\to+\infty} \frac{f(x_0+t_kd)-f(x_0)-t_k\nabla f(x_0)d}{t_k^2} = \frac{l'}{2},\ l' \leq \overline{f}''_P(x_0;d)$$

$$\lim_{k\to+\infty} \left|\frac{f(x_0+t_kd_k)-f(x_0+t_kd)-t_k\nabla f(x_0)(d_k-d)}{t_k^2}\right| =$$

$$\lim_{k\to+\infty} \left|\frac{t_k\nabla f(\xi_k)(d_k-d)-t_k\nabla f(x_0)(d_k-d)}{t_k^2}\right| \leq$$

$$\lim_{k\to+\infty} \frac{K\|d_k-d\|\|\xi_k-x_0\|}{t_k} = 0,$$

where $\xi_k \in [x_0+t_kd_k, x_0+t_kd]$ and then $\frac{\xi_k-x_0}{t_k} \to d$. Therefore:

$$\lim_{k\to+\infty} \frac{f(x_0+t_kd_k)-f(x_0+t_kd)-t_k\nabla f(x_0)(d_k-d)}{t_k^2} = 0$$

and then $l' = l$. Then $\overline{f}''_H(x_0;d) \leq \overline{f}''_P(x_0;d)$. The proof of the second equality is analogous. □

Theorem 2.1. *Let f be a function of class $C^{1,1}$ at x_0. Then:*

i) $\overline{f}''_H(x_0;d) = \overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d,d) \leq \overline{f}''_Y(x_0;d,d) \leq \overline{f}''_C(x_0;d,d)$.

ii) $\overline{f}''_H(x_0;d) = \overline{f}''_P(x_0;d) \leq \overline{f}''_R(x_0;d,d) \leq \overline{f}''_Y(x_0;d,d) \leq \overline{f}''_C(x_0;d,d)$.

Proof. i) From the previous remarks, it is only necessary to prove the inequality $\overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d,d)$. If we take the functions $\phi_1(t) = f(x_0+td) - t\nabla f(x_0)d$ and $\phi_2(t) = t^2$, applying Cauchy's theorem, we obtain:

$$2\frac{f(x_0+td)-f(x_0)-t\nabla f(x_0)d}{t^2} = 2\frac{\phi_1(t)-\phi_1(0)}{\phi_2(t)-\phi_2(0)} =$$

$$2\frac{\phi_1'(\xi)}{\phi_2'(\xi)} = \frac{\nabla f(x_0+\xi d)d-\nabla f(x_0)d}{\xi},$$

where $\xi = \xi(t) \in (0,t)$, and then $\overline{f}''_P(x_0;d) \leq \overline{f}''_D(x_0;d,d)$.

ii) From the previous remarks, it is only necessary to prove the inequalities $\overline{f}''_P(x_0;d) \leq \overline{f}''_R(x_0;d,d) \leq \overline{f}''_Y(x_0;d,d)$. Concerning the first inequality, from the definition of $\overline{f}''_P(x_0;d)$ we have:

$$f(x_0+td) = f(x_0) + t\nabla f(x_0)d + \frac{t^2}{2}\overline{f}''_P(x_0;d) + g(t)$$

where:

$$\limsup_{t\downarrow 0} \frac{g(t)}{t^2} = 0$$

and:

$$f(x_0+2td) = f(x_0) + 2t\nabla f(x_0)d + 2t^2\overline{f}''_P(x_0;d) + g(2t)$$

where:

$$\limsup_{t\downarrow 0} \frac{g(2t)}{t^2} = 4\limsup_{t\downarrow 0} \frac{g(2t)}{4t^2} = 0.$$

Then:

$$\limsup_{t\downarrow 0} \frac{f(x_0+2td) - 2f(x_0+td) + f(x_0)}{t^2} =$$

$$\limsup_{t\downarrow 0} \frac{t^2\overline{f}''_P(x_0;d) + g(2t) - g(t)}{t^2} \geq$$

$$\overline{f}''_P(x_0;d) + \limsup_{t\downarrow 0} \frac{g(2t)}{t^2} - \limsup_{t\downarrow 0} \frac{g(t)}{t^2}.$$

Then $\overline{f}''_R(x_0;d,d) \geq \overline{f}''_P(x_0;d)$. For the second inequality, we define $\phi_1(t) = f(x_0+2td) - 2f(x_0+td)$ and $\phi_2(t) = t^2$. Then, by Cauchy's theorem, we obtain:

$$\frac{f(x_0+2td) - 2f(x_0+td) + f(x_0)}{t^2} = \frac{\phi_1(t) - \phi_1(0)}{\phi_2(t) - \phi_2(0)} =$$

$$\frac{\phi'_1(\xi)}{\phi'_2(\xi)} = \frac{\nabla f(x_0+2\xi d)d - \nabla f(x_0+t\xi d)d}{\xi},$$

where $\xi = \xi(t) \in (0,t)$, and then $\overline{f}''_R(x_0;d,d) \leq \overline{f}''_Y(x_0;d,d)$.

□

3 Optimality conditions for constrained optimization problems

The aim of this section is to give necessary and sufficient conditions for the following optimization problem:

$$OP) \qquad \min_{x\in S} f(x)$$

where S is the feasible region defined as:

$$S = \{x \in \mathbb{R}^n : h_i(x) = 0, i = 1\ldots m, g_j(x) \leq 0, j = 1\ldots r, x \in X\}.$$

We suppose that the functions $f : \mathbb{R}^n \to \mathbb{R}$, $h_i : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, are of class $C^{1,1}$, $i = 1 \dots m$, $j = 1 \dots r$ and X is a closed convex subset of $\mathbb{R}^n$. From these conditions one can easily deduce the results in [19, 28, 49].

The following definitions will be useful in order to obtain optimality conditions for OP).

Definition 3.1. *Let A be a subset of $\mathbb{R}^n$ and $x_0 \in \mathrm{clA}$, where* clA *denotes the closure of A. The sets:*

- *i)* $WF(A, x_0) = \{d : \exists t_k \downarrow 0, x_0 + t_k d \in A, \forall k\}$
- *ii)* $F(A, x_0) = \{d : \forall t_k \downarrow 0, x_0 + t_k d \in A, \forall k\}$
- *iii)* $T(A, x_0) = \{d : \exists d_k \to d, \exists t_k \downarrow 0, x + t_k d_k \in A\}$
- *iv)* $T^2(A, x_0, d) = \{w \in \mathbb{R}^n : \exists w_k \to w, \exists t_k \downarrow 0, x + t_k d + 2^{-1} t_k^2 w_k \in A\}$
- *v)* $T_0^2(A, x_0, d) = \{w \in \mathbb{R}^n : \exists w_k \to w, \exists t_k \downarrow 0, \exists \gamma_k \downarrow 0, \gamma_k^{-1} t_k^2 \to 0, x + t_k d + \gamma_k w_k \in A\}$

are called, respectively, the cone of weakly feasible directions of A at x_0, the cone of feasible directions of A at x_0, the contingent cone of A at x_0, the second order contingent set and the asymptotic second order contingent set of A at x_0 in the direction d.

Theorem 3.1 (Proposition 3.3.11, [3]). *Let x_0 be a local minimum point of the problem OP) where f, h_i, and g_j are of class C^1 from $\mathbb{R}^n$ to $\mathbb{R}$ and X is a closed convex set. Then there exists a scalar μ_0 and Lagrange multipliers λ_i and μ_j satisfying the following conditions:*

- *i) for all $x \in X$, we have:*

$$\left(\mu_0 \nabla f(x_0) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x_0) + \sum_{j=1}^{r} \mu_j \nabla g_j(x_0) \right) (x - x_0) \geq 0$$

- *ii)* $\mu_j \geq 0$, $j = 0 \dots r$
- *iii)* μ_0, λ_i *and* μ_j *are not all equal to* 0.
- *iv) in every neighborhood N of x_0 there is an $x \in X \cap N$ such that $\lambda_i h_i(x) > 0$ for all i with $\lambda_i \neq 0$ and $\mu_j g_j(x) > 0$ for all j with $\mu_j \neq 0$ (this condition implies complementary slackness $\mu_j g_j(x_0) = 0$.)*

Theorem 3.2 (Proposition 3.3.12, [3]). *Let x_0 be a local minimum of the problem OP) and assume that the following two conditions hold:*

- *a) there does not exist a nonzero vector $\lambda = (\lambda_1, \dots, \lambda_m)$ such that:*

$$\left(\sum_{i=1}^{m} \lambda_i \nabla h_i(x_0) \right) (x - x_0) \geq 0, \forall x \in X$$

- *b) there exists a feasible direction d of X at x_0 such that:*

$$\nabla h_i(x_0)d = 0, i = 1 \dots m, \nabla g_j(x_0)d < 0, \forall j \in A(x_0)$$

where $A(x_0) = \{j : g_j(x_0) = 0\}$.

Then the previous condition holds with $\mu_0 = 1$.

Given $\lambda \in \mathbb{R}^m$, $\mu_0 \in \mathbb{R}$ and $\mu \in \mathbb{R}^r$, let $L(x, \lambda, \mu)$ the usual Lagrangian function, that is:

$$L(x, \lambda, \mu) = \mu_0 f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{r} \mu_j g_j(x),$$

and $G(\mu) = \{x \in X : \mu g(x) = 0, h(x) = 0\}$. We will write $L(x)$ instead of $L(x, \lambda, \mu)$ when this doesn't create confusion. The function $L(\cdot, \lambda, \mu)$ is of class $C^{1,1}$.

Theorem 3.3 (Necessary optimality condition). *Let x_0 be a local minimum point and assume that the previous two conditions a) and b) are satisfied. Then:*

- *there exists $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^r$, $\mu_j \geq 0$, $j = 1 \dots r$, such that $\mu_j g_j(x_0) = 0$ and $\nabla_x L(x_0, \lambda, \mu)d \geq 0$, $\forall d \in T(G(\mu), x_0)$;*
- *$\forall d \in T(G(\mu), x_0)$ such that $\nabla_x L(x_0, \lambda, \mu)d = 0$, we have:*

$$\nabla_x L(x_0, \lambda, \mu)w + \overline{L_{x}}''_H(x_0, \lambda, \mu; d) \geq 0,$$

$\forall w \in T^2(G(\mu), x_0, d)$.

Proof.

- i) See theorem 3.1.
- ii) Suppose, ab absurdo, $\exists d \in T(G(\mu), x_0)$ with $\nabla_x L(x_0, \lambda, \mu)d = 0$, and $w \in T^2(G(\lambda), x_0, d)$ such that:

$$\nabla_x L(x_0, \lambda, \mu)w + \overline{L_{x}}''_H(x_0, \lambda, \mu; d) < 0.$$

Since $w \in T^2(G(\mu), x_0, d)$ then there exist $w_k \to w$, $t_k \downarrow 0$ such that $x_0 + t_k d + 2^{-1}t_k^2 w_k \in G(\mu)$. So, by extracting eventually subsequences, we have:

$$l = \lim_{k \to +\infty} 2\frac{L(x_0 + t_k(d + 2^{-1}t_k w_k)) - L(x_0) - t_k \nabla_x L(x_0)(d + 2^{-1}t_k w_k)}{t_k^2}$$

where $l \in \partial_H^2 L(x_0; d)$. We observe that:

$$L(x_0 + t_k(d + 2^{-1}t_k w_k)) - L(x_0) =$$

$$f(x_0 + t_k(d + 2^{-1}t_k w_k)) + \mu g(x_0 + t_k(d + 2^{-1}t_k w_k)) + \lambda h(x_0 + t_k(d + 2^{-1}t_k w_k)) -$$

$$f(x_0) - \mu g(x_0) - \lambda h(x_0) \geq 0.$$

Then:

$$0 \leq L(x_0 + t_k(d + 2^{-1}t_k w_k)) - L(x_0) =$$

$$t_k \nabla_x L(x_0)(d + 2^{-1} t_k w_k) + t_k^2 \frac{l}{2} + o(t_k^2) =$$

$$2^{-1} t_k^2 (\nabla_x L(x_0) w_k + l) + o(t_k^2).$$

Taking the limit when $n \to +\infty$, we have:

$$0 \leq \nabla_x L(x_0) w + l \leq \nabla_x L(x_0) w + \overline{L_x}''_H(x_0; d).$$

□

Corollary 3.1. *Let x_0 be a local minimum point for the problem:*

$$\min_{x \in X} f(x).$$

Then:

i) $\nabla f(x_0) d \geq 0$, $\forall d \in T(X, x_0)$.

ii) $\overline{f}''_H(x_0; d) \geq 0$, $\forall d \in WF(X; x_0)$, $\nabla f(x_0) d = 0$.

Proof. i) Trivial. ii) If $d \in WF(X, x_0)$, $\nabla f(x_0) d = 0$, then $0 \in T^2(X, x_0, d)$ and so $\overline{f}''_H(x_0; d) \geq 0$. □

Theorem 3.4 (Sufficient optimality condition). *Let $x_0 \in S$, $\mu \in \mathbb{R}^r$, $\mu_i \geq 0$, $\mu_i g_i(x_0) = 0$, $\lambda \in \mathbb{R}^m$, such that $\nabla_x L(x_0, \lambda, \mu) d \geq 0$, $\forall d \in T(S, x_0)$. Suppose that $\forall d \in T(S, x_0)$ such that $\nabla_x L(x_0, \lambda, \mu) d = 0$, we have:*

$$\nabla_x L(x_0) w + \underline{L_x}''_H(x_0, \lambda, \mu; d) > 0$$

$\forall w \in T^2(S, x_0, d)$, $wd = 0$ and:

$$\nabla_x L(x_0, \lambda, \mu) w > 0$$

$\forall w \in T_0^2(S, x_0, d)$, $w \neq 0$, $wd = 0$. Then x_0 is a strict local minimum point.

Proof. Ab absurdo, let $x_k \to x_0$, $x_k \in S$, be a sequence such that $f(x_k) \leq f(x_0)$, $x_k \neq x_0$. If $t_k = \|x_k - x_0\|$ and $d_k = t_k^{-1}(x_k - x_0)$ then, eventually by extracting subsequences, $t_k \downarrow 0$, $d_k \to d$ and $d \in T(S, x_0)$. So:

$$0 \geq f(x_k) + \mu g(x_k) + \lambda h(x_k) - f(x_0) - \mu g(x_0) - \lambda h(x_0) = L(x_k) - L(x_0)$$

and then $\nabla_x L(x_0) d \leq 0$. So $\nabla_x L(x_0) d = 0$. Now let $s_k = \|d_k - d\|$ and $w_k = s_k^{-1}(d_k - d)$. If $d_k \neq d$, $\forall n \in \mathbb{N}$ (eventually by extracting a subsequence) then $x_k = x_0 + t_k d + t_k s_k w_k$, $s_k \neq 0$ and $w_k \to w \neq 0$. Since $d_k, d \in S^1$, then:

$$\|d_k\|^2 = \|d\|^2 + \|s_k w_k\|^2 + 2 s_k w_k d$$

and then $w_k d = -2^{-1} s_k \|w_k\|^2$. Taking the limit when $k \to +\infty$, we have $wd = 0$. So, by $C^{1,1}$ regularity, we have:

$$l = 2 \lim_{k \to +\infty} \frac{L(x_0 + t_k d_k) - L(x_0) - t_k \nabla_x L(x_0) d_k}{t_k^2}$$

and then:

$$0 \geq L(x_k) - L(x_0) = t_k \nabla_x L(x_0)(d + s_k w_k) + t_k^2 \frac{l}{2} + o(t_k^2),$$

and so:

$$0 \geq 2 s_k t_k^{-1} \nabla_x L(x_0) w_k + l + \epsilon_k.$$

Now if $t_k^{-1} s_k \to r \geq 0$ then, taking the limit when $n \to +\infty$, we obtain:

$$0 \geq 2r \nabla L(x_0) w + \underline{L_x}''_H(x_0; d)$$

which contradicts the hypothesis, since $2rw \in T^2(S, x_0, d)$. If $t_k^{-1} s_k \to +\infty$ then $w \in T_0^2(S, x_0, d)$ and:

$$0 \geq \nabla L(x_0) w_k + t_k s_k^{-1} \frac{l}{2} + t_k s_k^{-1} o(t_k^2)$$

and so $\nabla_x L(x_0) w \leq 0$, which contradicts the hypothesis. If $d_k = d$, $\forall n \in \mathbb{N}$, then $d \in F(S, x_0)$ and then $0 \in T^2(S, x_0, d)$. Following the same calculations of the previous case, we obtain a contradiction. □

4 Final remarks on optimality conditions for $C^{1,1}$ vector optimization problems by scalarization

In this section we briefly discuss how to obtain optimality conditions for $C^{1,1}$ vector optimization problems by scalarization techniques. Many different ways to reduce a multi-objective problem to a single-objective ones have been proposed; we focus the attention on linear scalarization methods which preserve the $C^{1,1}$ regularity of the involved data. A vector function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class $C^{1,1}$ if and only if each component of f is of class $C^{1,1}$. Given $f : \mathbb{R}^n \to \mathbb{R}^s$, $g : \mathbb{R}^n \to \mathbb{R}^r$ and $h : \mathbb{R}^n \to \mathbb{R}^m$, and a given subset X of $\mathbb{R}^n$, consider the following optimization problems:

1) $$\min_{x \in S} f(x)$$

2) $$\min_{x \in S} \theta f(x)$$

where S, the feasible region, is defined as:

$$S = \{x \in \mathbb{R}^n : h_i(x) = 0, i = 1 \dots m, g_j(x) \leq 0, j = 1 \dots r, x \in X\}.$$

We remember the following definition of vector minimum point and weak vector minimum point of the problem 1).

Definition 4.1. *A point $\bar{x} \in S$ is said to be a vector minimum point of f if there is no $x \in S$, $x \neq \bar{x}$, such that $f(x) - f(\bar{x}) \in \mathbb{R}^m_-$.*

Definition 4.2. *A point $\bar{x} \in S$ is said to be a weak vector minimum point of f if there is no $x \in S$, such that $f(x) - f(\bar{x}) \in \mathrm{int}\mathbb{R}^m_-$.*

The following theorem gives a necessary and sufficient condition for vector minimum points.

Theorem 4.1. *[33] Let $\theta \in \text{int}\mathbb{R}^{\text{m}}_+$. Then, $\bar{x}$ is a vector minimum point of 1) if an only if it is a minimum point of the single objective problem:*

$$3) \qquad \min_{x \in S'} \theta f(x)$$

where $S' = \{x \in S : f(\bar{x}) - f(x) \in \mathbb{R}^m_+\}$.

A sufficient condition for vector weak minimum point is given in the following result.

Definition 4.3. *[33] The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be convexlike on $S \subset \mathbb{R}^n$ if and only if for any $x_1, x_2 \in S$ and any $t \in [0,1]$ there exists $x_3 \in S$ such that:*

$$tf(x_1) + (1-t)f(x_2) - f(x_3) \in \mathbb{R}^m_+$$

Theorem 4.2. *[33]*

- *Let $\theta \in \mathbb{R}^m_+$ and $\bar{x}$ be a minimum point of 2). If $\theta \neq 0$, then $\bar{x}$ is a vector weak minimum point of 1).*
- *Suppose that f is convexlike on S. If $\bar{x} \in S$ is a vector weak minimum point of 1), then there exists $\theta \in \mathbb{R}^m_+ \backslash \{0\}$ such that $\bar{x}$ is a minimum point of 2).*

Nonlinear scalarizations of vector problems don't preserve the $C^{1,1}$ regularity, that is starting from a $C^{1,1}$ vector function the scalarized vector function is not a $C^{1,1}$ function. Indeed, by linear scalarization, one can use the results of the previous sections for the problem 2) in order to obtain optimality conditions for the problem 1).

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On the Hyers-Ulam stability of a Difference Equation *

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Abstract

In this paper, we investigate the Hyers-Ulam stability problem for the difference equation $f(x+p,y+q)+\varphi(x,y)f(x,y)+\psi(x,y)=0$ and its applications.

Keywords : Hyers-Ulam stability; difference equation

AMS 2000 Subject classifications : Primary : 39A11, Secondary : 39B72

1 Introduction

In 1940, S. M. Ulam [16] raised a question concerning the stability of group homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If the answer is affirmative, we would call the equation of homomorphism $H(xy)=H(x)H(y)$ stable. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2–12, 14, 15].

Mathematical computations frequently are based on equations, called difference equations or recurrence equations that allows us to compute the value of a function recursively from a given set

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of values. These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields. The study of linear difference equations is important for a number of reactions. Many types of problems are naturally formulated as linear equations [1, 13].

In the next section we consider the following difference equation of the form

$$f(x+p, y+q) + \varphi(x,y)f(x,y) + \psi(x,y) = 0, \tag{1.1}$$

$$f(x+p, y+q) + \varphi(x,y)f(x,y) = 0 \tag{1.2}$$

with restricted conditions of $\varphi(x,y)$ and $\psi(x,y)$.

It is important to provide methods and suitable criterion that describe the nature and behavior of solutions of difference systems, without actually constructing or approximating them. In contrast with differential equations, since the existence and uniqueness of solutions of discrete initial value problems is already guaranteed, one of the problems is the study of asymptotic behavior of solutions of the difference system [1].

Apart from the stability of solutions of the difference equation, in this paper we examine the situations that the difference equation (1.1) is stable in the sense of Hyers and Ulam, i.e., for a given δ-approximate function we construct a true solution of the difference equation near it. Throughout this paper, let $\delta > 0$ and $p, q \in \mathbb{N}$ be fixed, and $\mathbb{N}$ denote the set of all positive integers and for some nonnegative integer k, $\mathbb{N}_k := \{k, k+1, k+2, \cdots\}$.

2 Main Results

Before taking up the main subject we point out the following situation which is similar to that of elementary homogeneous linear differential equation. That is, if a particular solution f_p of (1.1) is given, then the general solution f of (1.1) has the form $f = f_h + f_p$, where f_h is a solution of (1.2).

In the next theorem, let two functions $\varphi : \mathbb{N}_k \times \mathbb{N}_k \to (0, \infty)$, $\psi : \mathbb{N}_k \times \mathbb{N}_k \to \mathbb{R}$ satisfy

$$\varepsilon(x,y) := \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{\varphi(x+ip, y+iq)} < \infty, \tag{2.1}$$

$$\varepsilon'(x,y) := \sum_{j=0}^{\infty} \frac{(-1)^j \psi(x+jp, y+jq)}{\prod_{i=0}^{j} \varphi(x+ip, y+iq)} < \infty \tag{2.2}$$

for all $x, y \in \mathbb{N}_k$. We now investigate the Hyers-Ulam stability problem for the equation (1.1).

That is, the difference equation (1.1) is stable in the sense of Hyers and Ulam under the conditions subject to (2.1), (2.2).

Theorem 2.1 *Suppose that functions $f, \psi : \mathbb{N}_k \times \mathbb{N}_k \to \mathbb{R}$ and φ satisfy the inequality*

$$|f(x+p, y+q) + \varphi(x,y)f(x,y) + \psi(x,y)| \leq \delta \tag{2.3}$$

for all $x, y \in \mathbb{N}_k$. Then there exist unique functions $T, T_h, T_p : \mathbb{N}_k \times \mathbb{N}_k \to \mathbb{R}$ such that T, T_p satisfy the equation (1.1), T_h satisfies the equation (1.2) and the relations

$$\begin{aligned} &|f(x,y) - T(x,y)| \leq \delta\varepsilon(x,y), \\ &|f(x,y) - T_h(x,y)| \leq \delta\varepsilon(x,y) + |\varepsilon^{'}(x,y)|, \\ &|T_p(x,y)| \leq |\varepsilon^{'}(x,y)|, \\ &T(x,y) = T_h(x,y) + T_p(x,y) \end{aligned} \tag{2.4}$$

hold for all $x, y \in \mathbb{N}_k$. If $\liminf\{f(x,y)|x, y \in \mathbb{N}\} > 0$, then the range of T_h is in $(0, \infty)$.

Proof. We prove the theorem by dividing into various steps.

Step 1. We show that there exists a limit function

$$T(x,y) := \lim_{m\to\infty} \left\{ T_{2m}(x,y) := \frac{f(x+2mp, y+2mq)}{\prod_{i=0}^{2m-1} \varphi(x+ip, y+iq)} - \sum_{j=0}^{2m-1} (-1)^j \frac{\psi(x+jp, y+jq)}{\prod_{i=0}^{j} \varphi(x+ip, y+iq)} \right\}$$

for any $x, y \in \mathbb{N}_k$.

Replacing x, y by $x+p$, $y+q$, respectively, in (2.3), we have

$$|f(x+2p, y+2q) + \varphi(x+p, y+q)f(x+p, y+q) + \psi(x+p, y+q)| \leq \delta \tag{2.5}$$

for all $x, y \in \mathbb{N}_k$. Combining the last inequality with (2.3), we get the relation

$$\begin{aligned} |f(x+2p, y+2q) \quad &- \quad \varphi(x+p, y+q)\varphi(x,y)f(x,y) + \psi(x+p, y+q) \\ &- \quad \varphi(x+p, y+q)\psi(x,y)| \leq \delta + \delta\varphi(x+p, y+q) \end{aligned} \tag{2.6}$$

for all $x, y \in \mathbb{N}_k$. Now we use the induction on $m \in \mathbb{N}$ to prove

$$\begin{aligned} \Big| f(x+2mp, y+2mq) &- \prod_{i=0}^{2m-1} \varphi(x+ip, y+iq)f(x,y) \\ &- \sum_{j=0}^{2m-1} (-1)^j \psi(x+jp, y+jq) \prod_{i=j+1}^{2m-1} \varphi(x+ip, y+iq) \Big| \\ &\leq \delta \sum_{j=0}^{2m-1} \prod_{i=j+1}^{2m-1} \varphi(x+ip, y+iq), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Big|f(x+(2m+1)p,y+(2m+1)q)+\prod_{i=0}^{2m}\varphi(x+ip,y+iq)f(x,y) \\ +\sum_{j=0}^{2m}(-1)^j\psi(x+jp,y+jq)\prod_{i=j+1}^{2m}\varphi(x+ip,y+iq)\Big| \\ \le\delta\sum_{j=0}^{2m}\prod_{i=j+1}^{2m}\varphi(x+ip,y+iq)\end{aligned} \tag{2.8}$$

for all $x,y\in\mathbb{N}_k$, where $\prod_i^j(\cdot)=1$ conveniently if $i>j$. The validity of (2.7) and (2.8) is easily proved by joining it to the other together with (2.3). Thus we obtain the inequalities which play an important role in the sequel,

$$\begin{aligned}\Big|\frac{f(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}-f(x,y)-\sum_{j=0}^{2m-1}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}\Big| \\ \le\delta\sum_{j=0}^{2m-1}\frac{1}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\Big|\frac{f(x+(2m+1)p,y+(2m+1)q)}{\prod_{i=0}^{2m}\varphi(x+ip,y+iq)}+f(x,y)+\sum_{j=0}^{2m}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}\Big| \\ \le\delta\sum_{j=0}^{2m}\frac{1}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}.\end{aligned} \tag{2.10}$$

We claim that the even sequence

$$\left\{T_{2m}(x,y)=\frac{f(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}-\sum_{j=0}^{2m-1}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}\right\} \tag{2.11}$$

is a Cauchy sequence. Indeed, for $m>n$ we get from (2.9)

$$\begin{aligned}|T_{2m}(x,y)-T_{2n}(x,y)|=\frac{1}{\prod_{i=0}^{2n-1}\varphi(x+ip,y+iq)}\cdot\Big|\frac{f(x+2mp,y+2mq)}{\prod_{i=2n}^{2m-1}\varphi(x+ip,y+iq)} \\ -f(x+2np,y+2nq)-\sum_{j=2n}^{2m-1}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=2n}^{j}\varphi(x+ip,y+iq)}\Big| \\ \le\frac{\delta}{\prod_{i=0}^{2n-1}\varphi(x+ip,y+iq)}\sum_{j=2n}^{2m-1}\frac{1}{\prod_{i=2n}^{j}\varphi(x+ip,y+iq)} \\ \to 0\quad\text{as}\quad n\to\infty.\end{aligned} \tag{2.12}$$

Therefore, we can now define a function $T:\mathbb{N}_k\times\mathbb{N}_k\to\mathbb{R}$ by

$$T(x,y)=\lim_{m\to\infty}T_{2m}(x,y) \tag{2.13}$$

for any $x, y \in \mathbb{N}_k$.

Step 2. We claim that T satisfies the equation (1.1).

We denote the odd sequence in (2.10) by

$$\left\{ S_{2m+1}(x,y) = \frac{f(x+(2m+1)p, y+(2m+1)q)}{\prod_{i=0}^{2m}\varphi(x+ip, y+iq)} + \sum_{j=0}^{2m}(-1)^j \frac{\psi(x+jp, y+jq)}{\prod_{i=0}^{j}\varphi(x+ip, y+iq)} \right\}. \quad (2.14)$$

Replacing x, y by $x+2mp$, $y+2mq$, respectively, in (2.3) and dividing by $\prod_{i=0}^{2m}\varphi(x+ip, y+iq)$, we obtain

$$|S_{2m+1}(x,y) + T_{2m}(x,y)| \le \frac{\delta}{\prod_{i=0}^{2m}\varphi(x+ip, y+iq)}, \quad (2.15)$$

which implies that a function S given by

$$S(x,y) = \lim_{m\to\infty} S_{2m+1}(x,y) \quad (2.16)$$

is defined since the right hand side in (2.15) tends to 0 as $m \to \infty$, and so $S(x,y) + T(x,y) = 0$ for any $x, y \in \mathbb{N}_k$ by (2.15).

Since $T_{2m}(x+p, y+q) = \varphi(x,y)S_{2m+1}(x,y) - \psi(x,y)$, we have

$$T(x+p, y+q) = \varphi(x,y)S(x,y) - \psi(x,y) = -\varphi(x,y)T(x,y) - \psi(x,y) \quad (2.17)$$

for any $x, y \in \mathbb{N}_k$. That is, T is a solution of the equation (1.1).

Step 3. We conclude that there exists a limit

$$T_h(x,y) := \lim_{m\to\infty} \left\{ \frac{f(x+2mp, y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip, y+iq)}, \right\}$$

which satisfies the equation (1.2) for any $x, y \in \mathbb{N}_k$.

From (2.9) we obtain that for any $x, y \in \mathbb{N}_k$

$$\begin{aligned} &\left| \frac{f(x+2mp, y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip, y+iq)} - f(x,y) \right| \quad (2.18)\\ &\quad \le \delta \sum_{j=0}^{2m-1} \frac{1}{\prod_{i=0}^{j}\varphi(x+ip, y+iq)} + \left| \sum_{j=0}^{2m-1} (-1)^j \frac{\psi(x+jp, y+jq)}{\prod_{i=0}^{j}\varphi(x+ip, y+iq)} \right|. \end{aligned}$$

Also from (2.10) we have that for any $x, y \in \mathbb{N}_k$

$$\begin{aligned} &\left| \frac{f(x+(2m+1)p, y+(2m+1)q)}{\prod_{i=0}^{2m}\varphi(x+ip, y+iq)} + f(x,y) \right| \quad (2.19)\\ &\quad \le \delta \sum_{j=0}^{2m} \frac{1}{\prod_{i=0}^{j}\varphi(x+ip, y+iq)} + \left| \sum_{j=0}^{2m} \frac{(-1)^j \psi(x+jp, y+jq)}{\prod_{i=0}^{j}\varphi(x+ip, y+iq)} \right|. \end{aligned}$$

Using the similar argument to that of (2.12), we obtain that the sequence

$$\left\{\frac{f(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}\right\} \tag{2.20}$$

is a Cauchy sequence due to (2.18), and the function T_h given by

$$T_h(x,y)=\lim_{m\to\infty}\left\{\frac{f(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}\right\} \tag{2.21}$$

is defined for any $x,y\in\mathbb{N}_k$.

As in the case of (2.15), we have from (2.3)

$$\begin{aligned}\Big|\frac{f(x+(2m+1)p,y+(2m+1)q)}{\prod_{i=0}^{2m}\varphi(x+ip,y+iq)}+\frac{f(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}\\ +\frac{\psi(x+2mp,y+2mq)}{\prod_{i=0}^{2m}\varphi(x+ip,y+iq)}\Big|\le\frac{\delta}{\prod_{i=0}^{2m}\varphi(x+ip,y+iq)},\end{aligned} \tag{2.22}$$

which yields that a function S_h given by

$$S_h(x,y)=\lim_{m\to\infty}\left\{\frac{f(x+(2m+1)p,y+(2m+1)q)}{\prod_{i=0}^{2m}\varphi(x+ip,y+iq)}\right\}$$

is defined and $S_h(x,y)+T_h(x,y)=0$ by (2.22), and hence

$$T_h(x+p,y+q)=\varphi(x,y)S_h(x,y)=-\varphi(x,y)T_h(x,y)$$

for any $x,y\in\mathbb{N}_k$. That is, T_h is a solution of the equation (1.2).

Step 4. We obtain that $T_p(x,y):=T(x,y)-T_h(x,y)$ is a particular solution of the equation (1.1) and T,T_p and T_h satisfy the inequalities (2.4).

By the comment preceding the theorem,

$$T_p(x,y):=T(x,y)-T_h(x,y)=-\sum_{j=0}^{\infty}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)} \tag{2.23}$$

is well defined and a particular solution of the equation (1.1).

We also have the inequalities (2.4) by taking the limit on both sides in (2.9) and (2.18).

Step 5. We prove the uniqueness of T,T_p and T_h subject to (2.4).

Now assume that T',T_h',T_p' are another mappings satisfying the conclusions in the theorem. Since T,T' satisfy the equation (1.1),

$$T(x,y)=\frac{T(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}-\sum_{j=0}^{2m-1}(-1)^j\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}$$

and

$$T^{'}(x,y)=\frac{T^{'}(x+2mp,y+2mq)}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}-\sum_{j=0}^{2m-1}(-1)^{j}\frac{\psi(x+jp,y+jq)}{\prod_{i=0}^{j}\varphi(x+ip,y+iq)}$$

for any $x,y\in\mathbb{N}_k$ and thus it then follows from (2.4) that

$$\begin{aligned}|T(x,y)-T^{'}(x,y)| &= \frac{|T(x+2mp,y+2mq)-T^{'}(x+2mp,y+2mq)|}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)} \\ &\leq \frac{2\delta}{\prod_{i=0}^{2m-1}\varphi(x+ip,y+iq)}\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{1}{\varphi(x+2mp+ip,y+2mq+iq)} \\ &\to 0 \quad \text{as} \quad m\to\infty.\end{aligned} \tag{2.24}$$

This implies the uniqueness of T. Similarly we have the uniqueness of T_h by the same method of (2.24). This completes the proof of the theorem. □

Remark 2.2 *If $\psi=0$, then $T=T_h, T_p=0$ since $\varepsilon^{'}(x,y)=0$ for any $x,y\in\mathbb{N}_k$.*

In the next theorem, let two functions $\varphi:\mathbb{N}_k\to(0,\infty)$, $\psi:\mathbb{N}_k\to\mathbb{R}$ satisfy

$$\varepsilon(x):=\sum_{j=0}^{\infty}\prod_{i=0}^{j}\frac{1}{\varphi(x+ip)}<\infty, \tag{2.25}$$

$$\varepsilon^{'}(x):=\sum_{j=0}^{\infty}\frac{(-1)^{j}\psi(x+jp)}{\prod_{i=0}^{j}\varphi(x+ip)}<\infty \tag{2.26}$$

for all $x\in\mathbb{N}_k$. Then we obtain the Hyers-Ulam stability problem for a single variable.

Theorem 2.3 *Suppose that functions $f,\psi:\mathbb{N}_k\to\mathbb{R}$ and φ satisfy the inequality*

$$|f(x+p)+\varphi(x)f(x)+\psi(x)|\leq\delta$$

for all $x\in\mathbb{N}_k$. Then there exist unique functions $T,T_h,T_p:\mathbb{N}_k\to\mathbb{R}$ such that T,T_p satisfy the equation $f(x+p)+\varphi(x)f(x)+\psi(x)=0$, T_h satisfies the equation $f(x+p)+\varphi(x)f(x)=0$ and the relations

$$\begin{aligned}&|f(x)-T(x)|\leq\delta\varepsilon(x),\\ &|f(x)-T_h(x)|\leq\delta\varepsilon(x)+|\varepsilon^{'}(x)|,\\ &|T_p(x)|\leq|\varepsilon^{'}(x)|,\\ &T(x)=T_h(x)+T_p(x)\end{aligned}$$

hold for all $x\in\mathbb{N}_k$.

3 Applications

We list some examples of difference equations which are stable by Theorem 2.1 in the sense of Hyers and Ulam.

The function $f(x) = \int_1^e (lnt)^x dt$ $(x \in \mathbb{N}_0)$ is a solution of the nonhomogeneous difference equation

$$f(x+1) + (x+1)f(x) - e = 0. \tag{3.1}$$

We obtain from Theorem 2.3 the Hyers-Ulam stability for a single variable as follows.

Corollary 3.1 *Suppose that a function $f : \mathbb{N}_0 \to \mathbb{R}$ satisfies the inequality*

$$|f(x+1) + (x+1)f(x) - e| \leq \delta \tag{3.2}$$

for all $x \in \mathbb{N}_0$. Then there exist unique functions $T, T_h, T_p : \mathbb{N}_0 \to \mathbb{R}$ such that T, T_p satisfy the equation (3.1), T_h satisfies the equation $f(x+1) + (x+1)f(x) = 0$ and the relations

$$\begin{aligned} &|f(x) - T(x)| \leq (e-1)\delta, \quad |f(x) - T_h(x)| \leq (e-1)\delta + e, \\ &|T_p(x)| \leq e, \quad T(x) = T_h(x) + T_p(x) \end{aligned} \tag{3.3}$$

hold for all $x \in \mathbb{N}_0$.

Proof. We apply Theorem 2.3 with $\varphi(x) = x+1$ and $\psi(x) = -e$. For any $x \in \mathbb{N}_0$

$$\begin{aligned} \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{x+i+1} &= \frac{1}{x+1} + \frac{1}{(x+1)(x+2)} + \cdots \\ &\leq e-1 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \left| \sum_{j=0}^{\infty} \frac{(-1)^j(-e)}{\prod_{i=0}^{j} x+i+1} \right| &= \frac{e}{x+1}\left(1 - \frac{1}{x+2} + \frac{1}{(x+2)(x+3)} + \cdots\right) \\ &\leq e. \end{aligned} \tag{3.5}$$

Thus we lead to the conclusion. □

The function $f(x) = \int_0^1 t^x e^{t-1} dt$ is a solution of the nonhomogeneous difference equation

$$f(x+1) + (x+1)f(x) - 1 = 0 \quad (x \in \mathbb{N}_0). \tag{3.6}$$

We obtain from Theorem 2.3 the Hyers-Ulam stability of (3.6), of which the proof is similar to that of Corollary 3.1.

Corollary 3.2 *Suppose that a function $f : \mathbb{N}_0 \to \mathbb{R}$ satisfies the inequality*

$$|f(x+1) + (x+1)f(x) - 1| \leq \delta \tag{3.7}$$

for all $x \in \mathbb{N}_0$. Then there exist unique functions $T, T_h, T_p : \mathbb{N}_0 \to \mathbb{R}$ such that T, T_p satisfy the equation (3.6), T_h satisfies the equation $f(x+1) + (x+1)f(x) = 0$ and the relations

$$\begin{aligned} &|f(x) - T(x)| \leq (e-1)\delta, \quad |f(x) - T_h(x)| \leq (e-1)\delta + 1, \\ &|T_p(x)| \leq 1, \quad T(x) = T_h(x) + T_p(x) \end{aligned} \tag{3.8}$$

hold for all $x \in \mathbb{N}_0$.

The function $f(x) = \int_0^1 \frac{t^x}{5+t} dt$ $(x \in \mathbb{N})$ is a solution of the nonhomogeneous difference equation

$$f(x+1) + 5f(x) - \frac{1}{x+1} = 0. \tag{3.9}$$

We obtain from Theorem 2.3 the Hyers-Ulam stability of (3.9) as follows.

Corollary 3.3 *Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ satisfies the inequality*

$$\left| f(x+1) + 5f(x) - \frac{1}{x+1} \right| \leq \delta \tag{3.10}$$

for all $x \in \mathbb{N}$. Then there exist unique functions $T, T_h, T_p : \mathbb{N} \to \mathbb{R}$ such that T, T_p satisfy the equation (3.9), T_h satisfies the equation $f(x+1) + 5f(x) = 0$ and the relations

$$\begin{aligned} &|f(x) - T(x)| \leq \frac{\delta}{4}, \quad |f(x) - T_h(x)| \leq \frac{\delta}{4} + \frac{1}{10}, \\ &|T_p(x)| \leq \frac{1}{10}, \quad T(x) = T_h(x) + T_p(x) \end{aligned} \tag{3.11}$$

hold for all $x \in \mathbb{N}$.

Proof. We apply Theorem 2.3 with $\varphi(x) = 5$ and $\psi(x) = \frac{-1}{x+1}$. For any $x \in \mathbb{N}$

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{5} = \frac{1}{5} + \frac{1}{5^2} + \cdots \leq \frac{1}{4} \tag{3.12}$$

and

$$\left| \sum_{j=0}^{\infty} \frac{(-1)^j \frac{-1}{x+j+1}}{\prod_{i=0}^{j} 5} \right| = \frac{1}{5} \left(\frac{1}{x+1} - \frac{1}{5(x+2)} + \cdots \right) \leq \frac{1}{10}. \tag{3.13}$$

This leads to the conclusion. □

The function $f(x) = \int_1^\infty \frac{\sqrt{3+6t}}{t^x} dt$ $(x \in \mathbb{N}_2)$ is a solution of the nonhomogeneous difference equation

$$f(x+1) + \frac{2x-3}{x} f(x) - \frac{9}{x} = 0. \tag{3.14}$$

Here we denote $\kappa := \sum_{j=0}^{\infty} \frac{(j+2)!}{\prod_{i=0}^{j} 2i+1}$ for the sake of abbreviation. We obtain the Hyers-Ulam stability of (3.14) from Theorem 2.3.

Corollary 3.4 *Suppose that a function $f : \mathbb{N}_2 \to \mathbb{R}$ satisfies the inequality*

$$\left| f(x+1) + \frac{2x-3}{x} f(x) - \frac{9}{x} \right| \leq \delta \tag{3.15}$$

for all $x \in \mathbb{N}_2$. Then there exist unique functions $T, T_h, T_p : \mathbb{N}_2 \to \mathbb{R}$ such that T, T_p satisfy the equation (3.14), T_h satisfies the equation $f(x+1) + \frac{2x-3}{x} f(x) = 0$ and the relations

$$\begin{aligned} &|f(x) - T(x)| \leq \kappa\delta, \quad |f(x) - T_h(x)| \leq \kappa\delta + 9, \\ &|T_p(x)| \leq 9, \quad T(x) = T_h(x) + T_p(x) \end{aligned} \tag{3.16}$$

hold for all $x \in \mathbb{N}_2$.

Proof. We apply Theorem 2.3 with $\varphi(x) = \frac{2x-3}{x}$ and $\psi(x) = \frac{-9}{x}$. For any $x \in \mathbb{N}_2$

$$\begin{aligned} \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{x+i}{2x+2i-3} &= \frac{x}{2x-3} + \frac{x(x+1)}{(2x-3)(2x-1)} + \cdots \\ &\leq \kappa \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \left| \sum_{j=0}^{\infty} \frac{(-1)^j \frac{-9}{x+j}}{\prod_{i=0}^{j} \frac{2x+2i-3}{x+i}} \right| &= \frac{9}{2x-3} \left(1 - \frac{x}{2x-1} + \frac{x(x+1)}{(2x-1)(2x+1)} + \cdots \right) \\ &\leq 9. \end{aligned} \tag{3.18}$$

This yields the conclusion. □

The function $f(x) = \int_1^\infty \frac{1}{t^x \sqrt{3+6t}}\ dt$ $(x \in \mathbb{N})$ is a solution of the nonhomogeneous difference equation

$$f(x+1) + \frac{2x-1}{x} f(x) - \frac{1}{x} = 0. \tag{3.19}$$

Here we denote $\omega = \sum_{j=1}^{\infty} \frac{j!}{\prod_{i=1}^{j} 2i-1}$ for abbreviation. We obtain from Theorem 2.3 the Hyers-Ulam stability as follows. The proof of Corollary 3.5 is similar to that of Corollary 3.4.

Corollary 3.5 *Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ satisfies the inequality*

$$|f(x+1) + \frac{2x-1}{x} f(x) - \frac{1}{x}| \leq \delta \tag{3.20}$$

for all $x \in \mathbb{N}$. Then there exist unique functions $T, T_h, T_p : \mathbb{N} \to \mathbb{R}$ such that T, T_p satisfy the equation (3.19), T_h satisfies the equation $f(x+1) + \frac{2x-1}{x} f(x) = 0$ and the relations

$$|f(x) - T(x)| \leq \omega\delta, \quad |f(x) - T_h(x)| \leq \omega\delta + 1, \tag{3.21}$$
$$|T_p(x)| \leq 1, \quad T(x) = T_h(x) + T_p(x)$$

hold for all $x \in \mathbb{N}$.

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ON SOME ESTIMATIONS OF KRAFT NUMBERS

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Abstract. In this paper, we give some results on some estimations of Kraft numbers and related results. Our results improve some results proved by N. M. Dragomir et al. [1].

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Key Words and Phrases: Kraft's inequality and number, Jensen's inequality and r-ary code with codeword length.

1. Introduction

The following two theorems are important in coding theory (see, for instance, [1], [4, p. 43–47]):

Theorem 1.1. (Kraft's Theorem) *We have the following:*

(1) *If C is an r-ary instantaneous code with codeword lengths $l_1, l_2, \cdots, l_n$, then these lengths must satisfy Kraft's inequality:*

$$\sum_{k=1}^{n} \frac{1}{r^{l_k}} \leq 1. \tag{1.1}$$

(2) *If the numbers $l_1, l_2, \cdots, l_n$ and r satisfying Kraft's inequality* (1.1)*, then there is an instantaneous r-ary code with codeword lengths $l_1, l_2, \cdots, l_n$.*

Theorem 1.2. (McMillan's Theorem) *If $C = \{c_1, c_2, \cdots, c_n\}$ is a uniquely decipherable r-ary code, then its codeword lengths must satisfy Kraft's inequality* (1.1).

Let us consider, for an r-ary code C having the codeword lengths $l_1, l_2, \cdots, l_n$, the Kraft numbers

$$K_r(l_1, l_2, \cdots, l_n) = \sum_{k=1}^{n} \frac{1}{r^{l_k}} \qquad (r > 1).$$

The following estimates of these numbers are given in [1]:

Theorem 1.3. *Let $C = \{c_1, c_2, \cdots, c_n\}$ be an r-ary code having the codeword lengths $l_1, l_2, \cdots, l_n$. Then we have the following inequalities:*

$$\begin{aligned} &\frac{1}{n \ln r} \sum_{k=1}^{n} \left[\ln(nr) - l_i (\ln r)^2\right] \\ &\le K_r(l_1, l_2, \cdots, l_n) \\ &\le \frac{1}{n \ln r} \sum_{k=1}^{n} \left[\frac{r^{l_i} \ln r + n \ln n - n l_i (\ln r)^2}{r^{l_i}}\right]. \end{aligned}$$

The equality holds if and only if $l_i = \log_r n$.

In this paper, we shall give some further improvements of Theorem 1.3 as well as some related results.

II. The Main Results

First, let us prove the following:

Theorem 2.1. *Let $C = (c_1, c_2, \cdots, c_n)$ be an r-ary code having the codeword lengths $l_1, l_2, \cdots, l_n$. Then we have the following inequalities:*

$$\begin{aligned} n r^{-\frac{1}{n}\sum_{i=1}^{n} l_i} &\le K_r(l_1, l_2, \cdots, l_n) \\ &\le n r^{-(\sum_{i=1}^{n} l_i r^{-c_i})/(\sum_{i=1}^{n} r^{-l_i})}. \end{aligned} \tag{2.1}$$

The equality holds if and only if $l_1 = l_2 = \cdots = l_n$.

Proof. We shall use the well-known Jensen inequality for convex functions $f : I \to \mathbb{R}$, that is,

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \le \frac{1}{n}\sum_{i=1}^{n} f(x_i), \tag{2.2}$$

where I is an interval of $\mathbb{R}$ and $x_i \in I$ for $i = 1, 2, \cdots, n$.

For $f(x) = r^x$, if $x_i = -l_i$ for all $i = 1, 2, \cdots, n$, then we have the first ineequality in (2.1). Note also that the function $f(x) = x \log_r x$ is convex since $r > 1$. Thus the inequality (2.2) for $x_i = r^{-l_i}$ $(i = 1, 2, \cdots, n)$ becomes the following inequalities:

$$\frac{1}{n}\sum_{i=1}^{n} r^{-c_i} \log_r \left(\frac{1}{n}\sum_{i=1}^{n} r^{-c_i} \right) \leq -\frac{1}{n}\sum_{i=1}^{n} l_i r^{-l_i},$$

i.e.,

$$\log_r \left(\frac{1}{n}\sum_{i=1}^{n} r^{-c_i} \right) \leq -\frac{\sum_{i=1}^{n} l_i r^{-l_i}}{\sum_{i=1}^{n} r^{-c_i}}$$

and so we have the second inequality in (2.1). This completes the proof.

As consequences of Theorem 2.1, we have the following corollaries from [1]:

Corollary 2.2. *Let $C = (c_1, c_2, \cdots, c_n)$ be an r-ary code having the codeword lengths $l_1, l_2, \cdots, l_n$. If we have the following inequalities:*

$$\frac{1}{n}\sum_{i=1}^{n} l_i < \log_r n, \tag{2.3}$$

then C is not uniquely decipherable.

Corollary 2.3. *If the real numbers r and l_i $(i = 1, 2, \cdots, n)$ satisfy the following inequality:*

$$\frac{\sum_{i=1}^{n} l_i r^{-l_i}}{\sum_{i=1}^{n} r^{-c_i}} \geq \log_r n, \tag{2.4}$$

then there is an instantaneous r-ary code with codeword lengths $l_1, l_2, \cdots, l_n$.

An extension of Theorem 1.3 is given by the following:

Theorem 2.4. *Let $C = (c_1, c_2, \cdots, c_n)$ be an r-ary code having the codeword lengths $l_1, l_2, \cdots, l_n$. For arbitrary $y, z \in \mathbb{R}$, we have the following inequalities:*

$$F(y) \leq K_r(l_1, l_2, \cdots, l_n) \leq G(z), \tag{2.5}$$

where

$$F(y) = nr^y - r^y \ln r \left(\sum_{i=1}^{n} l_i + ny \right), \tag{2.6}$$

$$G(z) = nr^z - \sum_{i=1}^{n} r^{-l_i}(l_i + z) \ln r. \tag{2.7}$$

The strongest inequalities of this form (2.5)*, that is,*

$$\begin{aligned} F(\tilde{y}) = \max_{y} F(y) &\le K_r(l_1, l_2, \cdots, l_n) \\ &\le G(\tilde{z}) = \min_{z} G(z), \end{aligned} \tag{2.8}$$

are obtained for

$$\tilde{y} = -\frac{1}{n}\sum_{i=1}^{n} l_i, \quad \tilde{z} = \log_r\left(\frac{1}{n}\sum_{i=1}^{n} r^{-c_i}\right) \tag{2.9}$$

Proof. Let us start, as in N. M. Dragomir et al. [1], from the following inequalities for a differentiable and convex function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$,

$$f'(y)(x-y) \le f(x) - f(y) \le f'(x)(x-y)$$

for all $x, y \in D(f)$ (: the domain of f). Then, letting $f(x) = r^x$ in the first inequality and $f(x) = r^x$, $y = z$, in the second inequality, then we have

$$r^y + r^y(x-y)\ln r \le r^x \le r^2 + r^x(x-z)\ln r. \tag{2.10}$$

Also setting $x = -l_i$ $(i = 1, 2, \cdots, n)$ and adding all such inequalities, we have

$$F(y) \le \sum_{i=1}^{n} r^{-l_i} \le G(z). \tag{2.11}$$

Further we have also

$$F'(y) = -r^y(\ln r)^2\left(\sum_{i=1}^{n} l_i + ny\right),$$

$$F''(y) = (\ln r)^2\left\{-r^y \ln r\left(\sum_{i=1}^{n} l_i + ny\right) - r^y\right\}.$$

From $F'(y) = 0$, it follows that $y = \tilde{y}$, while $F''(\tilde{y}) = -r^y(\ln r)^2 < 0$ and so we have the maximum. Similarly, we have also

$$G'(y) = nr^z \ln r - \ln r \sum_{i=1}^{n} r^{-l_i},$$

$$G''(y) = nr^z(\ln r)^2.$$

From $G'(z) = 0$, it follows that the minimum value is obtained for

$$nr^{\tilde{z}} = \sum_{i=1}^{n} r^{-l_i},$$

that is, for $z = \tilde{z}$. This completes the proof.

Remark 2.1. Note that the first inequality in (2.8) is equivalent to the first inequality in (2.1) and so the following improvement of the first inequality in (2.5) is valid:

$$F(y) \leq nr^{-\frac{1}{n}\sum_{i=1}^{n} l_i} \leq K_r(l_1, l_2, \cdots, l_n). \tag{2.12}$$

Moreover, this result is a simple consequence of the first inequality in (2.1) and the first inequality in (2.10). Indeed, setting

$$x = -\frac{1}{n}\sum_{i=1}^{n} l_i$$

in the first inequality in (2.10), we have

$$r^y - r^y\left(\frac{1}{n}\sum_{i=1}^{n} l_i + y\right)\ln r \leq r^{-\frac{1}{n}\sum_{i=1}^{n} l_i},$$

which is equivalent to the first inequality in (2.12).

For $y = z = \log_r\left(\frac{1}{n}\right)$, (2.12) together with the second inequality in (2.5) gives the following inequalities:

$$\begin{aligned} &\frac{1}{n\ln r}\sum_{i=1}^{n}\left[\ln(nr) - l_i(\ln r)^2\right] \\ &\leq nr^{-\frac{1}{n}\sum_{i=1}^{n} l_i} \\ &\leq K_r(l_1, l_2, \cdots, l_n) \\ &\leq \frac{1}{n\ln r}\sum_{i=1}^{n}\left[\frac{r^{l_i}\ln r + n\ln n - nl_i(\ln r)^2}{r^{l_i}}\right]. \end{aligned} \tag{2.13}$$

Remark 2.2. The second inequality in (2.8), after some elementary transformations, gives the second inequality in (2.1). The upper bounds for $K_r(l_1, l_2, \cdots, l_n)$ are nontrivial if they are ≤ 1. For example, for (2.1), we should have

$$nr^{-(\sum_{i=1}^{n} l_i r^{-l_i})/(\sum_{i=1}^{n} r^{-l_i})} \leq 1, \tag{2.14}$$

while, for (2.13), we should have

$$\frac{1}{n\ln r}\sum_{i=1}^{n}\left[\frac{r^{l_i}\ln r + n\ln n - nl_i(\ln r)^2}{r^{l_i}}\right] \leq 1. \tag{2.15}$$

The first condition, (2.14) and (2.15) give (2.4), which is the conclusion of Corollary 2.3.

Since the expression on the right-hand side of (2.4) is not simple one, we shall give some simple results. First, we prove the following:

Theorem 2.5. *Let the assumptions of Theorem 2.1 be satisfied and let*

$$m \le l_i \le M \qquad (i = 1, 2, \cdots, n). \tag{2.16}$$

Then we have the following inequality:

$$K_r(l_1, l_2, \cdots, l_n) \le n r^{-\frac{1}{K^2 n}\sum_{i=1}^{n} l_i}, \tag{2.17}$$

where

$$K = \frac{\sqrt{me^{-m}} + \sqrt{Me^{-M}}}{\sqrt{me^{-M}} + \sqrt{Me^{-m}}}. \tag{2.18}$$

Proof. Note that the following result is valid (see, for instance, [2, p. 307]):

If $0 < a \le a_i \le A$ and $0 < b \le b_i \le B$ for all $i = 1, 2, \cdots, n$, then we have

$$\frac{1}{\tilde{K}^2} \le \frac{n\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i} \le \tilde{K}^2, \tag{2.19}$$

where

$$\tilde{K} = \frac{\sqrt{ab} + \sqrt{AB}}{\sqrt{aB} + \sqrt{Ab}} \quad (\ge 1).$$

Setting $a_i = l_i$, $b_i = r^{-l_i}$ $(i = 1, 2, \cdots, n)$, $a = m$, $A = M$, $b = r^{-M}$ and $B = r^{-m}$, then the first inequality in (2.19) gives the following inequality:

$$\frac{\sum_{i=1}^{n} l_i r^{-l_i}}{\sum_{i=1}^{n} r^{-l_i}} \ge \frac{1}{K^2}\frac{1}{n}\sum_{i=1}^{n} l_i. \tag{2.20}$$

Therefore, from the second inequality in (2.1), we obtain (2.17). This completes the proof.

Corollary 2.6. *If the real numbers r and l_i $(i = 1, 2, \cdots, n)$ such that $0 < m \le l_i < M$ and $r > 1$ satisfy the following inequality:*

$$\frac{1}{n}\sum_{i=1}^{n} l_i \ge K^2 \log_r n, \tag{2.21}$$

where K is defined by (2.18), then there is an instantaneous r-ary code with codeword lengths $l_1, l_2, \cdots, l_n$.

Proof. The inequalities (2.20) and (2.21) gives the conclusion (2.4).

Theorem 2.7. *Let the assumptions of Theorem 2.5 be satisfied. Then we have the following inequalities:*

$$K_r(l_1, l_2, \cdots, l_n) \leq \frac{nM - \sum_{i=1}^n l_i}{M-m} r^{-m} + \frac{\sum_{i=1}^n l_i - nm}{M-m} r^{-M}. \tag{2.22}$$

Proof. The following result for a convex function $f : I \to \mathbb{R}$ is valid [3, p. 98]: If $m \leq x_i \leq M$ for $m, M \in I$ and $i = 1, 2, \cdots, n$, then we have

$$\frac{1}{n}\sum_{i=1}^n f(x_i) \leq \frac{M - \frac{1}{n}\sum_{i=1}^n x_i}{M-m} f(m) + \frac{\frac{1}{n}\sum_{i=1}^n x_i - m}{M-m} f(M). \tag{2.23}$$

Letting $f(x) = r^x, x_i = -l_i$ $(i = 1, 2, \cdots, n)$, $M \to -m$ and $m \to -M$, then we obtain (2.22). This completes the proof.

Corollary 2.8. *If the real numbers r and l_i $(i = 1, 2, \cdots, n)$ such that $0 < m \leq l_i < M$ and $r > 1$ satisfy the following inequality:*

$$\frac{1}{n}\sum_{i=1}^n l_i \geq \frac{Mr^{-m} - mr^{-m}}{r^{-m} - r^{-M}} - \frac{1}{n}\frac{M-m}{r^{-m} - r^{-M}}, \tag{2.24}$$

then there is an instantaneous r-ary code with codeword lengths $l_1, l_2, \cdots, l_n$.

Proof. It is easy to see that (2.24) is equivalent to the following:

$$\frac{nM - \sum_{i=1}^n l_i}{M-m} r^{-m} + \frac{\sum_{i=1}^n l_i - nm}{M-m} r^{-M} \leq 1$$

and so (2.22) gives

$$K_r(l_1, l_2, \cdots, l_n) \leq 1.$$

Applying Kraft's theorem, we get the above result. This completes the proof.

Theorem 2.9. *Let the assumptions of Theorem 2.1 be satisfied. Then we have the following inequality:*

$$\begin{aligned} &K_r(l_1, l_2, \cdots, l_n) \\ &\leq nr^{-\frac{1}{n}\sum_{i=1}^n l_i} - \ln r \sum_{i=1}^n l_i r^{-l_i} + \frac{\ln r}{n}\sum_{i=1}^n l_i \sum_{i=1}^n r^{-l_i} \end{aligned} \tag{2.25}$$

If $m \leq l_i \leq M$ for all $i = 1, 2, \cdots, n$, then we have the following inequality:

$$\begin{aligned} &K_r(l_1, l_2, \cdots, l_n) \\ &\leq nr^{-\frac{1}{n}\sum_{i=1}^n l_i} + \frac{\ln r}{n}\left[\frac{n^2}{4}\right](M-m)(r^{-m} - r^{-M}). \end{aligned} \tag{2.26}$$

Proof. The inequality (2.25) is a special case of the second inequality in (2.5) for $z = -\frac{1}{n}\sum_{i=1}^{n} l_i$. For the proof of (2.26), we need the following results ([2, p. 299], [3, p. 206]): Let a and b be n-tuples such that $u \le a_i \le U$ and $v \le b_i \le V$ for all $i = 1, 2, \cdots, n$ and real numbers u, v, U, V. Then we have the following inequality:

$$\left|\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \le \frac{1}{n}\left[\frac{n^2}{4}\right](U-u)(V-v). \tag{2.27}$$

Note that, for $a_i = l_i$ and $b_i = r^{-c_i}$ $(i = 1, 2, \cdots, n)$, we have $U = M$, $u = m$, $V = r^{-m}$, $v = r^{-M}$ and, by Čebysev's inequality, for oppositely ordered n-tuples $\{l_i\}$ and $\{v^{-l_i}\}$, we have

$$\left|\sum_{i=1}^{n} l_i r^{-l_i} - \sum_{i=1}^{n} l_i \sum_{i=1}^{n} r^{-l_i}\right| = \frac{1}{n}\sum_{i=1}^{n} l_i \sum_{i=1}^{n} r^{-l_i} - \sum_{i=1}^{n} l_i r^{-l_i}.$$

Therefore, (2.27) becomes the following inequality:

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} l_i \sum_{i=1}^{n} r^{-l_i} - \sum_{i=1}^{n} l_i r^{-l_i}\\ &\quad\le \frac{1}{n}\left[\frac{n^2}{4}\right](M-m)(r^{-m} - r^{-M})\end{aligned}$$

and, from (2.25), we have (2.26). This completes the proof.

Corollary 2.10. *If the real numbers $r > 1$ and l_i $(i = 1, 2, \cdots, n)$ satisfy the following inequality:*

$$nr^{-\frac{1}{n}\sum_{i=1}^{n} l_i} \le 1 + \ln r \sum_{i=1}^{n} \ln r^{-l_i} - \frac{\ln r}{n}\sum_{i=1}^{n} l_i \sum_{i=1}^{n} r^{-l_i}$$

or if the real numbers l_i $(i = 1, 2, \cdots, n)$ are such that $m \le l_i \le M$ and satisfy the following inequality:

$$nr^{-\frac{1}{n}\sum_{i=1}^{n} l_i} \le 1 - \frac{\ln r}{n}\left[\frac{n^2}{4}\right](M-m)(r^{-m} - r^{-M}),$$

then there is an instantaneous r-ary code with codeword lengths $l_1, l_2, \cdots, l_n$.

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Approximation of State-Dependent Impulsive Ordinary Differential Equations

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Abstract. In this work we consider approximations to the solution of a system of ordinary differential equations (ODE) in presence of possibly infinitely many state dependent impulses on the righthand side. We use a simple transformation that allows us to show that the problem is equivalent to an ordinary differential equation without impulse with its solution in $H^1(0,T;E)$. This transformation is numerically important. Indeed, to approximate the solution of the initial problem, we just have to approximate the solution of the equivalent problem which is regular. We then obtain better rates of convergence. Moreover we obtain an approximation of the solution which looks like the initial solution. Both, the solution and its approximation, are in $BV(0,T;E)$, the space of functions of bounded variation on $[0,T]$.

Keywords : ordinary differential equation, impulse, Galerkin method, numerical scheme.

AMS (MOS) subject classification : 65L05.

1 Introduction

In this paper we consider approximations to the solution of a system of ordinary differential equations (ODE) in presence of possibly infinitely many state dependent impulses on the righthand side [7], [8]:

$$\begin{cases} \dot{x}(t) = f(x(t),t) + \sum\limits_{j\in J} \alpha_j(x(\tau_j^-))\delta(t-\tau_j), & t \in [0,T]\,, \\ x(0) = x^0, \end{cases} \tag{1}$$

where $T > 0$ is a real number , $x^0 \in E$ is the initial condition, where $E = \mathbb{R}^d$ with d a positive integer, $x : [0,T] \longrightarrow E$ is a vector function , $f : E \times [0,T] \longrightarrow E$ is a given map, $\delta(.)$ is the Dirac delta distribution at 0, $J \subseteq \mathbb{N}^* = \{1,2,3,...\}$ is a countable set of indices, $\alpha_j : E \longrightarrow E$ is a given map for all $j \in J$ and $\{\tau_j\}_{j\in J}$ is a strictly increasing sequence on $]0,T]$.

In [6], the variational formulation is applied to obtain existence and uniqueness of the solution of (1). Then, using a Galerkin method with piecewise polynomial functions of degree K , it is proved that the rate of convergence in the L^2 -norm is $h^{1/2}$ and the rate of convergence at the nodes is h^1 . In the state independent impulses case the rate of convergence at the nodes grows up to h^{K+2} [3]. Moreover, when the solution is in $H^{K+1}(0,T;E)$, the rate of convergence in the L^2-norm and at the nodes become respectively h^{K+1} and h^{2K+2} [2] , [5].

In this paper we use a simple transformation that allows us to show that the problem is equivalent to an ordinary differential equation without impulse with its solution in $H^1(0,T;E)$. This transformation is numerically important. Indeed, to approximate the solution of (1), we just have to

approximate the solution of the equivalent problem which is regular. We then obtain better rates of convergence. Moreover we obtain an approximation of the solution which looks like the solution of (1). Both, the solution and its approximation, are in $BV(0,T;E)$, the space of functions of bounded variation on $[0,T]$.

In section 2, we present the equivalent problem without impulses (1). Its variational formulation is given in section 3, and we show existence and uniqueness of the solution under weaker assumptions than those required in [6]. In section 4, we consider the approximation problem by using piecewise polynomial functions. Then, in section 5, we show that the L^2 and the nodal convergence rates are h^1. Moreover, in the state-independent impulse case, we show that the nodal convergence rate is h^2 for nonlinear systems and h^{K+2} for linear systems. Numerical implementation is considered in section 6 and examples are given in section 7.

2 Transformation of the problem into an ordinary differential equation without impulse

A solution x of (1) is a function of bounded variation which is the sum of two parts: $x(t) = x_R(t) + x_J(t)$, a continuous part x_R in $H^1(0,T;E)$ (the Sobolev space of L^2-function with L^2-derivative) $x_R(t) = x^0 + \int_0^t f(x(\tau),\tau)d\tau$, and a jump part x_J in $L^\infty(0,T;E)$ (the space of essentially bounded functions) where $x_J(t) = \sum_{j\in J} \alpha_j(x(\tau_j^-))\chi_{[\tau_j,+\infty)}(t)$.

We introduce two families of functions: $\{\theta_j\}_{j\in J}$ and $\{\beta_j\}_{j\in J}$ from $\mathcal{C}(0,T;E)$ to E defined by

$$\begin{cases} \theta_1(y) = y(\tau_1), \\ \theta_j(y) = y(\tau_j) + \sum_{i<j} \alpha_i \circ \theta_i(y) \qquad \text{for } j > 1, \end{cases}$$

and $\beta_j = \alpha_j \circ \theta_j$ for $j \in J$. If x is the solution of (1), then $\alpha_j(x(\tau_j^-)) = \beta_j(x)$, and x_R is the solution of the following ODE

$$\begin{cases} \dot{x}_R(t) = F(x_R,t), \qquad t \in [0,T]\,, \\ x_R(0) = x^0, \end{cases} \tag{2}$$

where $F : \mathcal{C}(0,T;E) \times [0,T] \longrightarrow E$ and $F(y,t) = f\big(y(t) + \sum_{j\in J} \beta_j(y)\chi_{[\tau_j,+\infty)}(t), t\big)$.

3 Weak Formulation

Using the Galerkin method we approach the solution of (2) by a piecewise polynomial function u_h . The values $u_h(\tau_j)$ approach $x_R(\tau_j)$ $(j \in J)$. Then we show that $u_h + \sum_{j\in J} \beta_j(u_h)\chi_{[\tau_j,+\infty)}$ converges to the solution of (1).

The (global) weak variational formulation of (2) is

$$\begin{cases} \text{find } (x_R, X) \in H^1(0,T;E) \times E \text{ such that} \\ X \cdot v(T) - \int_0^T x_R(\tau) \cdot \dot{v}(\tau)d\tau = x^0 \cdot v(0) + \int_0^T F(x_R,\tau) \cdot v(\tau)d\tau, \\ \text{for all } v \in H^1(0,T;E). \end{cases} \tag{3}$$

Theorem 3.1. *Problems* (2) *and* (3) *are equivalent.*

Proof : See [3,Theorem 3.3]. □

A partition of the interval $[0,T]$ of size h is characterized by an integer $N \geq 1$ and a sequence $\{t_n\}_{n=0}^N$ of real numbers such that $0 = t_0 < \cdots < t_n < \cdots < t_N = T$, with $h = \max\{t_n - t_{n-1} : n =$

$1, 2, \ldots, N\}$. Subintervals will be denoted by $I_n =]\, t_{n-1}, t_n[\ ,\ n = 1, \ldots, N$. A family of partitions indexed by h is said to be *regular* if there exists a constant $c > 0$ such that as h goes to zero $c\,h \leq t_n - t_{n-1} \leq h$, $n = 1, \ldots, N$. Throughout this paper we shall only consider regular families of partitions.

For any $n = 1, ..., N$, let $\mathcal{U}_n = E^{n+1} \times \prod_{k=1}^{n} H^1(I_k; E)$ and for $\tilde{u}_n = (U_0, ..., U_n; u_1, ..., u_n) \in \mathcal{U}_n$ we define

$$\hat{u}_n(t) = \begin{cases} U_k & \text{if } t = t_k \quad (k = 0, ..., n), \\ u_k(t) & \text{if } t \in]t_{k-1}, t_k[\quad (k = 1, ..., n). \end{cases}$$

To obtain a piecewise polynomial approximation of the solution, we consider the following variational formulation

$$\begin{cases} \text{If } U_0 = x^0, \text{ and } (u_k, U_k) \in H^1(I_k; E) \times E \text{ is given} \\ \text{for } k = 1, 2, ...n-1, \text{find } (u_n, U_n) \in H^1(I_n; E) \times E \text{ such that} \\ U_n \cdot v_n(t_n) - \int_{I_n} u_n(\tau) \cdot \dot{v}_n(\tau) d\tau = U_{n-1} \cdot v_n(t_{n-1}) + \int_{I_n} F(\hat{u}_n, \tau) \cdot v_n(\tau) d\tau, \\ \text{for all } v_n \in H^1(I_n; E). \end{cases} \tag{4}$$

If this problem has a solution, we obtain, as in the proof of Theorem 3.1, $U_n = u_n(t_n)$ and $U_{n-1} = u_n(t_{n-1})$. Thus, we can easily show that $\hat{u}_n(.)$ is continuous and belongs to $H^1(0, t_n; E)$.

Theorem 3.2. *Problems* (3) *and* (4) *are equivalent.*

Proof : See [3,Theorem 3.3]. □

Let us define

$$J_n = \{j \in J : \tau_j \in]t_{n-1}, t_n]\}, \qquad n = 1, ..., N,$$
$$J_n^{\leq} = \{j \in J : \tau_j \in]0, t_n]\}, \qquad n = 1, ..., N$$

and

$$l_\infty(E^J) = \{\xi \in E^J : \| \xi \|_\infty = sup\{|\xi_j| : j \in J\} < +\infty\}.$$

By the same way we define $l_\infty(E^{J_n})$ and $l_\infty(E^{J_n^{\leq}})$.

To show existence and uniqueness of the solution to (4) we use the next lemmas [8] . To simplify we suppose $J = \mathbb{N}^*$, however our results remain true for any subset J of $\mathbb{N}^*$. Let $\Lambda = \sum_{i=1}^{+\infty} \lambda_i$, and set $\prod_{i=m}^{n} \gamma_i = 1$ if $n < m$ and $\sum_{i=m}^{n} \gamma_i = 0$ if $n < m$.

Lemma 3.1. *Let* $\{\lambda_j\}_{j \in J}$ *be a sequence of positive real numbers , then*

$$1 + \sum_{i=1}^{+\infty} \lambda_i \prod_{k=1}^{i-1} (1 + \lambda_k) \leq e^{\Lambda}. \qquad \square$$

Let us remark that the functions $\theta_j(.)$ are well defined only if the values $y(\tau_j), j \in J$, are defined. For a sequence $\xi = \{\xi_j\}_{j \in J}$ we can define

$$\begin{cases} \theta_1(\xi) = \xi_1, \\ \theta_j(\xi) = \xi_j + \sum_{i<j} \alpha_i \circ \theta_i(\xi) \qquad \text{for } j > 1, \end{cases}$$

with, as previously defined, $\beta_j = \alpha_j \circ \theta_j$ for $j \in J$.

Lemma 3.2. *For any* ξ^1 *and* ξ^2 *in* $l_\infty(E^j)$*, we have*

$$\begin{aligned} |\beta_j(\xi^1) - \beta_j(\xi^2)| &\leq \lambda_j e^{\Lambda} \max\{|\xi_k^1 - \xi_k^2| : k = 1, ..., j\} \\ &\leq \lambda_j e^{\Lambda} \|\xi^1 - \xi^2\|_\infty. \qquad \square \end{aligned}$$

Remark 3.1. *If $x_1, x_2 \in C(0,T;E)$ and $\xi_j^k = x_k(\tau_j)$ $(k=1,2)$, then*

$$\begin{aligned} |\beta_j(\xi^1) - \beta_j(\xi^2)| &\leq \lambda_j e^{\Lambda} \max\{|x_1(\tau_k) - x_2(\tau_k)| : k = 1, ..., j\} \\ &\leq \lambda_j e^{\Lambda} \|x_1 - x_2\|_{\infty}. \end{aligned}$$ □

Lemma 3.3. *We have* $\sum_{j\in J} |\beta_j(0)| \leq e^{\Lambda} \sum_{j\in J} |\alpha_j(0)|$. □

Lemma 3.4. *Let $I = [\alpha, \beta]$.*
(i) The map

$$\begin{aligned} H^1(I;E) &\longrightarrow L^2(I;E) \times E \\ v &\longmapsto (\dot{v}, v(\beta)) \end{aligned}$$

is an isomorphism.
(ii) Let b be an arbitrary element of $H^1(I,E)^*$, the variational problem

$$\begin{cases} \text{find } (u,U) \in L^2(I;E) \times E \text{ such that} \\ U.v(\beta) - \int_{\alpha}^{\beta} u(t) \cdot \dot{v}(t) dt = b(v) \\ \text{for all } v \in H^1(I;E) \end{cases} \tag{5}$$

has a unique solution.

Proof : See [10]. □

Theorem 3.3. *Let $x^0 \in E$,*
(a) assume the map $f : E \times [0,T] \longrightarrow E$ verifies the following conditions :
(i) for all $x \in E$, the map $t \longrightarrow f(x,t)$ is (Lebesgue) measurable,
(ii) there exists $q \in L^2(0,T;\mathbb{R})$ such that, for any $x_1, x_2 \in L^2(0,T;E)$

$$| f(x_1(t),t) - f(x_2(t),t) | \leq q(t) | x_1(t) - x_2(t) | \text{ a.e.},$$

(iii) the map $t \longrightarrow f(0,t)$ belongs to $L^2(0,T;E)$;
(b) suppose also the functions α_j $(j \in J)$ verify the following conditions :
(iv) $\sum_{j\in J} |\alpha_j(0)| < +\infty$,
(v) there exists a sequence of positive real numbers $\{\lambda_j\}_{j\in J}$ such that $\sum_{j\in J} \lambda_j < +\infty$ and for all $j \in J$ and $x_1, x_2 \in E$

$$| \alpha_j(x_1) - \alpha_j(x_2) | \leq \lambda_j | x_1 - x_2 | .$$

Then
(1) the solution $(U_1, ..., U_N, u_1, ..., u_N) \in \mathcal{U}_N$ to (4) exists and is unique, moreover, $x_R = \hat{u}_N$ is the unique solution in $H^1(0,T;E)$ to the system (2), and
(2) the solution to (1) exists and is unique in $BV(0,T;E)$, moreover, we have
$x(t) = \hat{u}_N(t) + \sum_{j\in J} \beta_j(\hat{u}_N) \chi_{[\tau_j, +\infty)}(t)$.

Proof : Similar to the proof of Theorem 3.5 in [6]. See also [11]. □

Let us consider the Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$ defined by

$$\mathcal{U} = E^{N+1} \times \prod_{n=1}^{N} H^1(I_n;E) = \mathcal{U}_N \text{ and } \mathcal{V} = E \times \prod_{n=1}^{N} H^1(I_n;E) = \mathcal{V}_N.$$

In order to apply a Galerkin method we consider the following equivalent problem which is a global mesh-dependent variational formulation

$$\begin{cases} \text{find } \tilde{u} = (U_0, U_1, ..., U_N; u_1, ..., u_N) \in \mathcal{U} \text{ such that} \\ U_0(V_0 - v_1(t_0)) + \sum_{n=1}^{N} U_n(v_n(t_n) - v_{n+1}(t_n)) + U_N \cdot v_N(t_N) - \sum_{n=1}^{N} \int_{I_n} u_n(\tau) \cdot \dot{v}_n(\tau) d\tau \\ = x^0 \cdot V_0 + \sum_{n=1}^{N} [\int_{I_n} F(\hat{u}_n, \tau) \cdot v_n(\tau) d\tau], \\ \text{for all } \tilde{v} = (V_0, v_1, ..., v_N) \in \mathcal{V}. \end{cases} \tag{6}$$

4 Galerkin Approximation

Let us define the subspaces $\mathcal{U}^h$ and $\mathcal{V}^h$ of $\mathcal{U}$ and $\mathcal{V}$ by

$$\mathcal{U}^h = \left\{ \begin{array}{c} \tilde{u}_h = (U_0, U_1, .., U_N; u_1, ..., u_N) \in \mathcal{U} \\ \text{such that } u_n \in \mathcal{P}^K(I_n; E) \text{ for } n = 1, ...N \end{array} \right\}, \tag{7}$$

and

$$\mathcal{V}^h = \left\{ \begin{array}{c} \tilde{v}^h = (V_0, v_1, v_2, ..., v_N) \in \mathcal{V} \\ \text{such that } v_n \in \mathcal{P}^{K+1}(I_n; E); \text{ for } n = 1, ..., N \end{array} \right\}, \tag{8}$$

where $\mathcal{P}^K(I_n; E)$ is the set of polynomials of degree at most K on I_n. For each $n = 1, ..., N$ we set

$$\mathcal{U}_n^h = E^{n+1} \times \prod_{k=1}^{n} \mathcal{P}^K(I_k, E),$$

and for each $\tilde{u}_n = (U_0, ..., U_n; u_1, ..., u_n) \in \mathcal{U}_n^h$ we define

$$\hat{u}_n(t) = \left\{ \begin{array}{ll} U_k & \text{if } t = t_k \quad (k = 0, ..., n), \\ u_k(t) & \text{if } t \in]t_{k-1}, t_k[\quad (k = 1, ..., n). \end{array} \right. \tag{9}$$

The discrete problem associated to (6) is

$$\left\{ \begin{array}{l} \text{find } \tilde{u}^h = (U_0, ..., U_N; u_1, ..., u_N) \in \mathcal{U}^h \text{ such that } U_0 = x^0 \text{ and for } n = 1, ..., N \\ U_n \cdot v_n(t_n) - \int_{I_n} u_n(\tau) \cdot \dot{v}_n(\tau) d\tau = U_{n-1} \cdot v_n(t_{n-1}) + \int_{I_n} F(\hat{u}_n, \tau) \cdot v_n(\tau) d\tau, \\ \text{for all } \ v_n \in \mathcal{P}^{K+1}(I_n; E). \end{array} \right. \tag{10}$$

We will show that (10) has a unique solution for h small enough. Similar to Lemma 3.4 and Theorem 3.3 we have

Lemma 4.1. *Let $I_n = [t_{n-1}, t_n]$.*
(i) The map

$$\begin{array}{rcl} \mathcal{P}^{K+1}(I_n; E) & \longrightarrow & \mathcal{P}^K(I_n; E) \times E \\ v & \longmapsto & (-\dot{v}, v(t_n)) \end{array}$$

is an isomorphism.
(ii) Let b an arbitrary element of $\mathcal{P}^{K+1}(I_n; E)^*$, the variational problem

$$\left\{ \begin{array}{l} \textit{find } (u, U) \in \mathcal{P}^K(I_n; E) \times E \textit{ such that} \\ U.v(t_n) - \int_{I_n} u(t).\dot{v}(t) dt = b(v) \quad \textit{for all } v \in \mathcal{P}^{K+1}(I_n; E), \end{array} \right.$$

has a unique solution. □

Theorem 4.1. *Let the assumptions of Theorem* 3.3 *be verified, then the variational problem* (10) *has a unique solution.*

Proof : Let $U_0 = x^0$ and, on each interval I_n, we suppose that (u_k, U_k), $k = 1, ..., n-1$, are given. We look for $(u_n, U_n) \in \mathcal{P}^K(I_n; E) \times E$ such that

$$\left\{ \begin{array}{l} U_n \cdot v_n(t_n) - \int_{I_n} u_n(\tau) \cdot \dot{v}_n(\tau) d\tau = U_{n-1} \cdot v_n(t_{n-1}) + \int_{I_n} F(\hat{u}_n, \tau) \cdot v_n(\tau) d\tau \\ \text{for } \ v_n \in \mathcal{P}^{K+1}(I_n; E). \end{array} \right.$$

For all $u_n \in \mathcal{P}^K(I_n; E)$, the map $v \longrightarrow b_n(v; u_n) = U_{n-1}.v(t_{n-1}) + \int_{I_n} F(\hat{u}_n, \tau) \cdot v(\tau) d\tau$ is well defined, linear and continuous on $\mathcal{P}^K(I_n; E)$, thus $b_n(.; u_n) \in \mathcal{P}^{K+1}(I_n; E)^*$. Let $(u_n^0, U_n^0) \in \mathcal{P}^K(I_n; E) \times E$ be arbitrary and fixed. We construct the sequence $\{(u_n^m, U_n^m)\}_{m=0}^{+\infty}$ as follows : given $(u_n^m, U_n^m) \in \mathcal{P}^K(I_n; E) \times E$, $(u_n^{m+1}, U_n^{m+1}) \in \mathcal{P}^K(I_n; E) \times E$ is the solution of the problem

$$U_n^{m+1}.v_n(t_n) - \int_{I_n} u_n^{m+1}(\tau) \cdot \dot{v}_n(\tau) d\tau = b_n(v_n; u_n^m), \tag{11}$$

 for all $v_n \in \mathcal{P}^{K+1}(I_n;E)$. We show that $\{u_n^m\}_{m=0}^{+\infty}$ is a Cauchy sequence in $\mathcal{P}^K(I_n;E)$. We have

$$\begin{aligned}&\mid F(\hat{u}_n^{m+1},\tau)-F(\hat{u}_n^m,\tau)\mid\\&\quad\leq q(\tau)\left[\mid u_n^{m+1}(\tau)-u_n^m(\tau)\mid+\mid\sum_{j\in J}(\beta_j(\hat{u}_n^{m+1})\chi_{[\tau_j,+\infty)}(\tau)-\beta_j(\hat{u}_n^m)\chi_{[\tau_j,+\infty)}(\tau))\mid\right].\end{aligned}$$

For v_n satisfying $\dot{v}_n(t)=-\left(u_n^{m+1}-u_n^m\right)(t)$ in I_n, $v_n(t_n)=0$, and using Lemma 4.1 and (11), we show that

$$\|u_n^{m+1}-u_n^m\|_{0,n}\leq h_n^{1/2}\|q\|_{0,n}\left[\|u_n^m-u_n^{m-1}\|_{0,n}+h_n^{1/2}\Lambda e^{\Lambda}\|\zeta^m-\zeta^{m-1}\|_{\infty,n}\right]$$

with $\zeta^j=u_n(\tau_j), j\in J_n$, and $\left\|\zeta^m-\zeta^{m-1}\right\|_{\infty,n}=sup\{|u_n^m(\tau_j)-u_n^{m-1}(\tau_j)|; j\in J_n^{\leq}\}$. For all $j\in J_{n-1}^{\leq}$ we have $\hat{u}_n^m(\tau_j)=\hat{u}_n^{m-1}(\tau_j)$ since u_k, $k=1,...,n-1$, are fixed. By using the following inequalities [2]

$$\left\|\zeta^m-\zeta^{m-1}\right\|_{\infty,n}\leq h_n^{-1/2}[\left\|\zeta^m-\zeta^{m-1}\right\|_{0,n}+h_n\left\|\zeta^m-\zeta^{m-1}\right\|_{1,n}]$$

$$\left\|\zeta^m-\zeta^{m-1}\right\|_{1,n}\leq ch_n^{-1}\left\|\zeta^m-\zeta^{m-1}\right\|_{0,n},$$

we obtain $\left\|u_n^{m+1}-u_n^m\right\|_{0,n}\leq ch_n^{1/2}\left\|q\right\|_{0,n}\left\|u_n^m-u_n^{m-1}\right\|_{0,n}$. If h_n is small enough then $\{u_n^m\}_{m=0}^{+\infty}$ is a Cauchy sequence in $\mathcal{P}^K(I_n;E)$. Then $\{u_n^m\}_{m=0}^{+\infty}$ converge to $u_n\in\mathcal{P}^K(I_n;E)$. By continuity we obtain $-\int_{I_n}u_n(\tau)\cdot\dot{v}_n(\tau)d\tau=b_n(v_n;u_n)$ for all $v_n\in H^1(I_n;E)$, such that $v_n(t_n)=0$. If in (11) we take $v_n(t)=V\in E$, we obtain $U_n^{m+1}.V=b_n(V;u_n^m)$ for all V in E. Then $\{U_n^m\}_{m=0}^{+\infty}$ converge to $U_n\in E$ and $U_n.V=b_n(V;u_n)$ for all $V\in E$. Let $w_n(t)=v_n(t)-v_n(t_n)$ with $v_n\in H^1(I_n;E)$, we have $-\int_{I_n}u_n(\tau)\cdot\dot{w}_n(\tau)d\tau=b_n(w_n;u_n)$, since $w_n(t_n)=0$, and we obtain

$$-\int_{I_n}u_n(\tau)\cdot\dot{v}_n(\tau)d\tau=b_n(v_n;u_n)-b_n(v_n(t_n);u_n)=b_n(v_n;u_n)-U_n.v_n(t_n).$$

Then (u_n,U_n) is a solution of (10) on I_n. □

5 L^2 and nodal errors

5.1 Convergence results

Theorem 5.1. *Let $x\in H^{K+1}(0,T;E)$, then*

$$Inf\{\|x-\overline{u}_n\|_{0,n}:\overline{u}_n\in\mathcal{P}^K(I_n;E)\}\leq ch^{k+1}\|x^{K+1}\|_{0,n},\tag{12}$$

$$Inf\{\|x-\overline{u}^h\|_0:\overline{u}^h|_{I_n}=\overline{u}_n\in\mathcal{P}^K(I_n;E)\}\leq ch^{k+1}\|x^{K+1}\|_0.\qquad\square\tag{13}$$

Let us set $e_{0,n}=\|u_n-x_R\|_{0,n}$, $e_0=\|u_n-x_R\|_0$, $\overline{e}_{0,n}=\|\overline{u}_n-x_R\|_{0,n}$, $E_n=|U_n-x_R(t_n)|$, $e_{1,n}=\|u_n-x_R\|_{1,n}$, $\overline{e}_{1,n}=\|\overline{u}_n-x_R\|_{1,n}$, $e_{\infty,n}=sup\{|u_n(\tau_j)-x_R(\tau_j)|:j\in J_n\}$ and $e_{\infty,n^{\leq}}=sup\{|\hat{u}_n(\tau_j)-x_R(\tau_j)|:j\in J_n^{\leq}\}$ where $\overline{u}_n\in\mathcal{P}^K(I_n;E)$.

Theorem 5.2. *Suppose the assumptions of Theorem 3.3 are verified. If h is small enough we have*

$$\|\hat{u}_N-x_R(t)\|_0\leq c\ h^1\left\|x_R^{(1)}\right\|_0,\tag{14}$$

$$\max\{|U_n-x_R(t_n)|;n=0,1,..,N\}\leq ch^1\left\|x_R^{(1)}\right\|_0.\tag{15}$$

Proof : We use a proof by induction. Suppose that $E_k = |U_k - x_R(t_k)| = O(h)$, $k = 1, ..., n-1$, and $\|u - x_R\|_{0,[0,t_{n-1}]} = O(h)$. From the following problems on I_n

$$\left\{\begin{array}{l} \text{find } x_n \in H^1(I_n;E), \text{ with } x_1, ..., x_{n-1} \text{ already known } x_n = x_R|_{I_n}, \text{ such that} \\ x_R(t_n)\cdot v_n(t_n) - \int_{I_n} x_R(\tau)\cdot \dot{v}_n(\tau)d\tau = x_R(t_{n-1})\cdot v_n(t_{n-1}) + \int_{I_n} F(x_R,\tau)\cdot v_n(\tau)d\tau, \\ \text{for all } v_n \in H^1(I_n;E) \end{array}\right.$$

and

$$\left\{\begin{array}{l} \text{find } u_n \in \mathcal{P}^K(I_n;E) \text{ with } u_1, ..., u_{n-1} \text{ already known, such that} \\ U_n\cdot v_n(t_n) - \int_{I_n} u_n(\tau)\cdot \dot{v}_n(\tau)d\tau = U_{n-1}\cdot v_n(t_{n-1}) + \int_{I_n} F(\hat{u}_n,\tau)\cdot v_n(\tau)d\tau, \\ \text{for all } v_n \in \mathcal{P}^{K+1}(I_n;E), \end{array}\right.$$

we obtain

$$\begin{array}{c}[U_n - x_R(t_n)]\cdot v_n(t_n) - \int_{I_n}(u_n(\tau) - x_R(\tau))\cdot \dot{v}_n(\tau)d\tau = [U_{n-1} - x_R(t_{n-1})]\cdot v_n(t_{n-1}) \\ + \int_{I_n}[F(\hat{u}_n,\tau) - F(x_R,\tau)]\cdot v_n(\tau)d\tau. \end{array} \tag{16}$$

Take $v_n = U_n - x_R(t_n)$ in this equation. Since $e_{\infty,n} \leq h_n^{-1/2}[e_{0,n} + h_n e_{1,n}]$, see [2], we obtain the following estimates

$$E_n \leq E_{n-1} + (1+\Lambda_n e^{\Lambda_n})\|q\|_{0,n}e_{0,n} + \Lambda_n e^{\Lambda_n} h_n^1 e_{1,n}\|q\|_{0,n} + \Lambda_n e^{\Lambda_n}\|q\|_{0,n}h^{1/2}e_{\infty,n-1\leq}.$$

Moreover, $e_{1,n} \leq \|u_n - \overline{u}_n\|_{1,n} + \overline{e}_{1,n}$ and $\|u_n - \overline{u}_n\|_{1,n} \leq c h_n^{-1}\|u_n - \overline{u}_n\|_{0,n}$. Then we obtain

$$E_n \leq E_{n-1} + (1+\Lambda_n e^{\Lambda_n})\|q\|_{0,n}e_{0,n} + \Lambda_n e^{\Lambda_n} h_n^1\{c h_n^{-1}\|u_n - \overline{u}_n\|_{0,n} + \overline{e}_{1,n} + h^{-1/2}e_{\infty,n-1\leq}\}\|q\|_{0,n}. \tag{17}$$

By taking v_n such that $\dot{v}_n(\tau) = -(u_n - \overline{u}_n)(\tau)$, $t \in I_n$, and $v_n(t_n) = 0$ in (16), we obtain

$$\begin{array}{rcl} \|u_n - \overline{u}_n\|_{0,n} & \leq & \dfrac{1}{1-\theta_n}\left\{h_n^{1/2}E_{n-1} + (1+\theta_n)\overline{e}_{0,n} + h_n^{3/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\overline{e}_{1,n}\right\} \\ & + & \dfrac{1}{1-\theta_n}h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}e_{\infty,n-1\leq}, \end{array} \tag{18}$$

where $\theta_n = h_n^{1/2}\|q\|_{0,n}(1 + (1+c)\Lambda_n e^{\Lambda_n})$. By using this inequality for the L^2-error, we obtain

$$e_{0,n} \leq \frac{1}{1-\theta_n}\left[h_n^{1/2}E_{n-1} + 2\overline{e}_{0,n} + h_n^{3/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\overline{e}_{1,n} + h_n^1\|q\|_{0,n}\Lambda_n e^{\Lambda_n}e_{\infty,n-1\leq}\right].$$

By the same way for the nodal estimates, we use (18), then (17) becomes

$$\begin{array}{rcl} E_n & \leq & E_{n-1}\dfrac{1+\|q\|_{0,n}\Lambda_n e^{\Lambda_n}h^{1/2}}{1-\theta_n} \\ & & + \overline{e}_{0,n}\left[(1+\Lambda_n e^{\Lambda_n})\|q\|_{0,n} + \dfrac{1+\theta_n}{1-\theta_n}(\|q\|_{0,n}(1+\Lambda_n e^{\Lambda_n}) + \|q\|_{0,n}\Lambda_n e^{\Lambda_n})\right] \\ & & + \overline{e}_{1,n}\left[h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n} + h_n^{3/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\dfrac{\|q\|_{0,n}(1+\Lambda_n e^{\Lambda_n}) + c\|q\|_{0,n}\Lambda_n e^{\Lambda_n}}{1-\theta_n}\right] \\ & & + e_{\infty,n-1\leq}\Lambda_n e^{\Lambda_n}h_n^{1/2}\|q\|_{0,n}\left[1 + \dfrac{c'}{1-\theta_n}(1+\Lambda_n e^{\Lambda_n})h_n^{1/2}\|q\|_{0,n}\right]. \end{array}$$

For h small enough, the coefficient of $e_{\infty,n-1\leq}$ can be rearranged as follows

$$\begin{array}{rcl} \Lambda_n e^{\Lambda_n}h_n^{1/2}\|q\|_{0,n}\left[1 + \dfrac{c'}{1-\theta_n}(1+\Lambda_n e^{\Lambda_n})h_n^{1/2}\|q\|_{0,n}\right] & \leq & \Lambda_n e^{\Lambda_n}h_n^{1/2}\|q\|_{0,n}(1 + \dfrac{c'}{1-\theta_n}\theta_n), \\ & \leq & \dfrac{c'\theta_n}{1-\theta_n}. \end{array}$$

The coefficient of $\overline{e}_{0,n}$ is

$$\frac{(1+\Lambda_n e^{\Lambda_n})\|q\|_{0,n}}{1-\theta_n}\left\{(1-\theta_n)+(1+\theta_n)+c(1+\theta_n)\frac{\Lambda_n e^{\Lambda_n}}{1+\Lambda_n e^{\Lambda_n}}\right\}\leq(1+\Lambda_n e^{\Lambda_n})\|q\|_{0,n}\frac{(3+\theta_n)}{1-\theta_n}.$$

and the coefficient of $\overline{e}_{1,n}$ is

$$\frac{h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}}{1-\theta_n}\left[1-\theta_n+\theta_n h_n^{1/2}+ch_n^{1/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\right]\leq\frac{h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}}{1-\theta_n}\left[1+ch_n^{1/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\right].$$

If we take $\alpha_n=\frac{1+\theta_n}{1-\theta_n}$, $\beta_n=\frac{\|q\|_{0,n}[1+\Lambda_n e^{\Lambda_n}c]}{1-\theta_n}(3+\theta_n)$, $\gamma_n=\frac{1}{1-\theta_n}\{h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}(1+\|q\|_{0,n}h_n^{1/2}\Lambda_n e^{\Lambda_n}c)\}$ and $\delta_n=\frac{c'\theta_n}{1-\theta_n}$, we obtain $E_n\leq\alpha_n E_{n-1}+\beta_n\overline{e}_{0,n}+\gamma_n\overline{e}_{1,n}+\delta_n e_{\infty,n-1\leq}$. Step by step we show that $E_n\leq KE_0+K\sum_{j=0}^{n}\beta_{n-j}\overline{e}_{0,n-j}+K\sum_{j=0}^{n}\gamma_{n-j}\overline{e}_{1,n-j}+K\sum_{j=0}^{n}\delta_{n-j}e_{\infty,n-j\leq}$, where K is defined by $K=\prod_{i=1}^{n}\alpha_i\leq e^{2\sum_{i=1}^{n}\frac{\theta_i}{1-\theta_i}}$ with $\alpha_n=\frac{1+\theta_n}{1-\theta_n}=1+\frac{2\theta_n}{1-\theta_n}$. If h is small enough so that $\theta_l<1/2$ for all l in the set $\{1,...,N\}$, then $K\leq e^{4\sum_{i=1}^{n}\theta_i}<+\infty$. We also have $\sum_{j=0}^{n}\beta_{n-j}\overline{e}_{0,n-j}\leq(\sum_{j=0}^{n}\beta_{n-j}^2)^{1/2}\overline{e}_0$ where $\overline{e}_0=(\sum_{j=0}^{n}\overline{e}_{0,n-j}^2)^{1/2}$. By the same way, if we take $\overline{e}_1=(\sum_{j=0}^{n}\overline{e}_{1,n-j}^2)^{1/2}$, we obtain $\sum_{j=0}^{n}\gamma_{n-j}\overline{e}_{1,n-j}\leq(\sum_{j=0}^{n}\gamma_{n-j}^2)^{1/2}\overline{e}_1$. We show that

$$\sum_{j=0}^{n}\beta_{n-j}^2=\sum_{m=0}^{n}\beta_m^2\leq\max\{\frac{(3+\theta_l)^2}{(1-\theta_l)^2}:0\leq l\leq n\}\sum_{m=0}^{n}\|q\|_{0,m}^2(1+\Lambda_m e^{\Lambda_m}c)^2.$$

If h is small enough, such that $\theta_i\leq 1/2$, we have $\sum_{j=0}^{n}\beta_{n-j}^2\leq c(1+\Lambda e^{\Lambda}c)^2\|q\|_0^2$. Since $1+h_n^{1/2}\Lambda_n e^{\Lambda_n}c\|q\|_{0,n}\leq 1+\theta_n$, then $\gamma_n\leq h_n\frac{1}{1-\theta_n}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}(1+\theta_n)$, and we obtain

$$\sum_{j=0}^{n}\gamma_{n-j}^2\leq c\|q\|_{0,m}^2(\Lambda e^{\Lambda}c)^2h^2.$$

We also have $\sum_{j=0}^{n}\delta_{n-j}=\sum_{k=0}^{n}\delta_k=\sum_{k=0}^{n}\frac{c'\theta_k}{1-\theta_k}\leq 2c'\sum_{k=0}^{n}\theta_k\leq 2hc'\|q\|_0(1+c\Lambda e^{\Lambda})$. Finally

$$E_n\leq KE_0+Kc(1+\Lambda e^{\Lambda}c)\|q\|_0\overline{e}_0+c\|q\|_0(\Lambda e^{\Lambda}c)h^1\overline{e}_1+K2c'\|q\|_0h^1(1+c\Lambda e^{\Lambda})e_{\infty,n-j\leq}.$$

We have $e_{\infty,n}\leq h_n^{1/2}[e_{0,n}+h_n e_{1,n}]$, moreover $e_{1,n}\leq\overline{e}_{1,n}+ch_n^{-1}\|u_n-\overline{u}_n\|_{0,n}$, then

$$\begin{aligned}e_{\infty,n}&\leq h_n^{-1/2}e_{0,n}+h_n^{1/2}\overline{e}_{1,n}+ch_n^{-1/2}\|u_n-\overline{u}_n\|_{0,n},\\&\leq 2h_n^{-1/2}\overline{e}_{0,n}+h^{-1/2}\frac{(1+c)}{1-\theta_n}\Big\{h_n^{1/2}E_{n-1}+(1+\theta_n)\overline{e}_{0,n}+h_n^{3/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\overline{e}_{1,n}\\&+h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}e_{\infty,n-1\leq}\Big\},\\&\leq O(h^{1/2})+cE_{n-1}+O(h^1)+O(h^{3/2})+\frac{1}{1-\theta_n}h_n^{1/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}e_{\infty,n-1\leq}.\end{aligned}$$

On the other hand

$$e_{0,n}\leq\frac{1}{1-\theta_n}\left[h^{1/2}E_{n-1}+2\overline{e}_{0,n}+h_n^{3/2}\|q\|_{0,n}\Lambda_n e^{\Lambda_n}\overline{e}_{1,n}+h_n\|q\|_{0,n}\Lambda_n e^{\Lambda_n}e_{\infty,n-1\leq}\right]$$

then

$$e_0 \leq C\Big[\max_n E_n + 2\overline{e}_0 + h^{3/2}\max\{\|q\|_{0,n}\Lambda_n e^{\Lambda_n}, n=1,...,N\}e_1$$
$$+h_n^{1/2}\max\{\|q\|_{0,n}\Lambda_n e^{\Lambda_n}, n=1,...,N\}e_{\infty,n-1\leq}\Big].$$

Let us go back to our proof by induction. For $n=1$ we have $E_0 = U_0 - x^0 = 0$, then $E_1 \cong O(h)$, therefore the result is true for E_1 and $e_{0,1}$. Let us suppose that it is true up to $n-1$ and we show that it is true for n. Since $E_1 \cong O(h)$ and $e_{0,1} \cong O(h)$ then $e_{\infty,1} \leq O(h^{1/2})$. Consequently $e_{\infty,2} \leq O(h^{1/2})$. Finally $e_{\infty,l} \cong O(h^{1/2})$ for all $l \leq n$ and

$$E_n \leq C\big(E_0 + O(h^1) + O(h^{3/2})\big),$$
$$e_0 \leq [E_0 + O(h_n) + O(h^{3/2})]. \quad \square$$

Theorem 5.3. *Under the assumptions of Theorem* 3.3 *and if h is small enough, we have*

$$\|\hat{u}_N + \sum_{j\in J}\beta_j(\hat{u}_N)\chi_{[\tau_j,+\infty)} - x\|_0 \leq ch^1\|x_R^{(1)}\|_0, \tag{19}$$

$$\max\big\{\big|U_n + \sum_{j\in J}\beta_j(\hat{u}_N)\chi_{[\tau_j,+\infty)}(t_n) - x(t_n)\big|; n=0,1,..,N\big\} \leq ch^1\left\|x_R^{(1)}\right\|_0 \tag{20}$$

where x is the solution of (1).

Proof : We have $x(t) = x_R(t) + \sum_{j\in J}\beta_j(x_R)\chi_{[\tau_j,+\infty)}(t_n)$, then

$$\big\|\hat{u}_N + \sum_{j\in J}\beta_j(\hat{u}_N)\chi_{[\tau_j,+\infty)} - x\big\|_0 \leq \|\hat{u}_N - x_R\|_0 + \|\sum_{j\in J}[\beta_j(\hat{u}_N) - \beta_j(x_R)]\chi_{[\tau_j,+\infty)}\|_0.$$

Using the definition of the norm $\|.\|_0$, we obtain

$$\begin{aligned}\big\|\sum_{j\in J}[\beta_j(\hat{u}_N) - \beta_j(x_R)]\chi_{[\tau_j,+\infty)}\big\|_0 &= \left\{\sum_{n=1}^{N}\big\|\sum_{j\in J_n}[\beta_j(\hat{u}_N) - \beta_j(x_R)]\chi_{[\tau_j,+\infty)}\big\|_{0,n}^2\right\}^{1/2},\\ &\leq e^{\Lambda}e_{\infty,N\leq}\left\{\sum_{n=1}^{N}\big\|\sum_{j\in J_n}\lambda_j\big\|_{0,n}^2\right\}^{1/2}\\ &\leq h^{1/2}\Lambda e^{\Lambda}e_{\infty,N\leq}.\end{aligned}$$

From the results of the preceding theorem, we obtain (19). Finally (20) follows. $\square$

5.2 Superconvergence results for nonlinear systems

In this paragraph we suppose the impulses are state-independent, $\alpha_j(x(\tau_j^-)) = \alpha_j, j \in J$. In this case $\beta_j(x_R) = \alpha_j, j \in J$, and the assumption (b) of Theorem 3.3 becomes $\sum_{j\in J}|\alpha_j| < +\infty$. With this condition, (10) reduces to the following problem

$$\begin{cases} \text{find } \tilde{u}^h = (U_0,...,U_N;u_1,...,u_N) \in \mathcal{U}^h \text{ such that } U_0 = x^0 \text{ and for } n=1,...,N\\ U_n\cdot v_n(t_n) - \int_{I_n}u_n(\tau)\cdot\dot{v}_n(\tau)d\tau\\ \quad = U_{n-1}\cdot v_n(t_{n-1}) + \int_{I_n}f(u_n(\tau) + \sum_{j\in J}\alpha_j\chi_{[\tau_j,+\infty)}(\tau),\tau)\cdot v_n(\tau)d\tau,\\ \text{for all } v_n \in \mathcal{P}^{K+1}(I_n;E). \end{cases} \tag{21}$$

For nonlinear systems, if we use the results of Theorem 5.2 and consider additional conditions on F, we obtain higher rates of convergence.

426 **Theorem 5.4.** *If $A = (a_{ij})_{1\le i,j\le d}$, where $a_{ij} \in L^2(0,T;\mathbb{R})$ and $f \in L^2(0,T;E)$ then (i) the unique solution w of the system $\dot{w}(t) = A(t)w(t) + f(t)$, $w(0) = \alpha$, is in $H^1(0,T;E)$, (ii) the map $(x^0, f) \mapsto w$ from $E \times L^2(0,T;E)$ to $H^1(0,T;E)$ is linear and continuous, and (iii) there exists a constant c such that*

$$\|w\|_1 \le c\{|\alpha| + \|f\|_0\}.$$

Proof : See [4]. □

Lemma 5.1. *Let x be a function defined from $[0,T]$ to E, and let x_n^I be its Lagrange interpolation of degree 0 on each subinterval I_n. If $x \in H^1(I_n;E)$, we have*

$$\|x_n^I - x\|_{0,n} \le ch^1\|x^1\|_{0,n} \le ch^1\|x\|_{1,n}$$

and

$$\|x^I - x\|_0 \le ch^1\|x^1\|_0 \le ch^1\|x\|_1.$$

Proof : See [1]. □

Theorem 5.5. *Suppose the solution of (2) is in $H^1(0,T;E)$, the solution of (6) exists and is unique, and the following conditions hold:*
(i) the matrix $A(t) = F_x(x,t)$, such that $a_{ij}(t) = \frac{\partial F_i}{\partial x_j}(x,t)$, exists and its columns are in $L^2(0,T;E)$,
(ii) there exists a neighborhood V of the origin of E and a constant $B \ge 0$ such that for any $t \in [0,T]$ and $y \in x_R(t) + V$

$$|F(y,t) - F(x_R,t) - A(t)(y - x_R(t))| \le B|y - x_R(t)|^2.$$

There exist a constant c independent of h such that for h small enough

$$\max\{|U_n - x_R(t_n)| : n = 0,1,..,N\} \le c\,\|u - x_R\|_0\,[h^1 + \|u - x_R\|_0], \tag{22}$$

$$\max\left\{\left|U_n + \sum_{j\in J}\alpha_j\chi_{[\tau_j,+\infty)}(t_n) - x(t_n)\right| : n = 0,1,..,N\right\} \le c\,\|u - x_R\|_0\,[h^1 + \|u - x_R\|_0]. \tag{23}$$

Proof : Similar to Theorem 4.6 of [3], based on Theorem 5.4 and Lemma 5.1. □

5.3 Superconvergence results for linear systems

In this paragraph we suppose that $f(x(t),t) = A(t)x(t) + b(t)$ where $A(t)$ is a matrix of dimension $d \times d$ whose elements are measurable and bounded on $[0,T]$, and $b(t)$ and the columns of $A(t)$ belongs to $H^{K+1}(0,T;E)$.

Lemma 5.2. *Let $w \in H^{K+2}(I_n;E)$ such that $\dot{w}(t) + A^T w(t) = 0, w(t_n^-) = E_n$. Then*

$$\left\|w^{(l)}\right\|_{\infty,n} \le c\,|E_n| \quad (l = 0,...,K+1), \tag{24}$$

$$\left\|w^{(l)}\right\|_{0,n} \le ch^{1/2}\,|E_n| \quad (l = 0,...,K+2). \tag{25}$$

Moreover, let $v \in \mathcal{P}^K(I_n;E)$ such that $v(t) = E_n - \int_t^{t_n} P_K(\dot{w})(\tau)d\tau$, where P_K is the L^2-projection on $\mathcal{P}^K(I_n;E)$, then

$$\|v\|_{\infty,n} \le (1 + ch)\,|E_n|\,, \tag{26}$$

$$\|v - w\|_{\infty,n} \le ch^{K+2}\,|E_n|\,, \tag{27}$$

$$\|v - w\|_{0,n} \le ch^{K+5/2}\,|E_n|\,, \tag{28}$$

$$\|\dot{v} - \dot{w}\|_{0,n} \le ch^{K+3/2}\,|E_n|\,. \tag{29}$$

Proof : See [3], Lemma 5.7. □

Lemma 5.3. *The following inequality is verified*

$$\prod_{n=1}^{N}(1+ch_n^{1/2}\left\|q\right\|_{0,n})\leq exp(cT^{1/2}\left\|q\right\|_0). \tag{30}$$

Proof : See [3], Lemma 5.4. □

Theorem 5.6. *Under the assumptions of Theorem* 3.3 *and if the impulses are state-independent, then for h small enough and* $\overline{u}_n\in\mathcal{P}^{k+1}(I_n;E)$ *we have*

$$|U_n-x_R(t_n)|\leq\left[1+ch_n^{1/2}\left\|q\right\|_{0,n}\right]|U_{n-1}-x_R(t_{n-1})|+c\left\|q\right\|_{0,n}\left\|\overline{u}_n-x_R\right\|_{0,n}, \tag{31}$$

$$\left\|u_n-x_R\right\|_{0,n}\leq ch_n^{1/2}|U_{n-1}-x_R(t_{n-1})|+c\left\|\overline{u}_n-x_R\right\|_{0,n}. \tag{32}$$

Proof : See [2], Theorem 4.10. □

Remark 5.1. *When the elements of A are in* $H^1(0,T;E)$*, they are continuous and bounded, therefore q is bounded and for any interval* I_n*,* $\left\|q\right\|_{0,n}$ *is proportional to* $h^{1/2}$*.* □

Theorem 5.7. *Suppose that* $f(x(t),t)=A(t)x(t)+b(t)$ *where* $b(t)$ *and the columns of* $A(t)$ *are in* $H^{K+1}(0,T;E)$*, and* $\sum\limits_{j\in J}|\alpha_j|<+\infty$*. Then if h is small enough we have*

$$|U_n-x_R(t_n)|\leq(1+ch)\left[|U_{n-1}-x_R(t_{n-1})|+ch^{K+3/2}\left\|u_n-x_R\right\|_{0,n}\right].$$

Proof : Similar to the proof of Theorem 5.10 of [2]. □

As a consequence of Theorems 5.1, 5.6 and 5.7 and Lemma 5.3 we obtain the next superconvergence result.

Theorem 5.8. *Under the assumptions of Theorem* 5.7*, we have the following result*

$$|U_n-x_R(t_n)|\leq c\left\{\left|U_0-x^0\right|+h^{K+2}\left\|x_R^{(1)}\right\|_0\sum\nolimits_{j\in J}|\alpha_j|\right\}. \qquad \square$$

6 Numerical implementation

In this paragraph we suppose that $E=\mathbb{R}$, we shall also write $\mathcal{P}^K(I_n)$ and $\mathcal{P}^K(0,1)$ for $\mathcal{P}^K(I_n;E)$ and $\mathcal{P}^K([0,1];E)$. We shall construct basis for $\mathcal{P}^K(0,1)$ and $\mathcal{P}^{K+1}(0,1)$ and we deduce basis for $\mathcal{P}^K(I_n)$ and $\mathcal{P}^{K+1}(I_n)$, $n=1,..,N$. Consider a $(k+1)$-points quadrature formula of the form

$$\int_0^1\psi(\zeta)\,d\zeta=\sum_{k=1}^{K+1}a_k\psi(\eta_k), \tag{33}$$

where $0\leq\eta_1\leq...\leq\eta_l\leq\eta_{l+1}\leq...\leq\eta_{K+1}\leq1$, such that this formula is exact for polynomials of degree at most $2K+1$. Once the quadrature points have been obtained, we use them to construct bases for u_n in $P^K(I_n)$ and v_n in $P^{K+1}(I_n)$. Let us denote by $\{\phi_k:k=1,\ldots,K+1\}$ the Lagrange interpolating polynomials associated with the $K+1$ points $\{\xi_k\}_{k=1}^{K+1}$,

$$\phi_k(\xi)=\prod_{\substack{i=1\\i\neq k}}^{K+1}\frac{\xi-\xi_i}{\xi_k-\xi_i},\quad k=1,\ldots,K+1, \tag{34}$$

and let

$$\phi_{nk}(t)=\phi_k\Big(\frac{t-t_{n-1}}{h_n}\Big),\qquad t_{n-1}\leq t\leq t_n. \tag{35}$$

Then $\{\phi_{nk} : k = 1, \ldots, K+1\}$ is the desired basis of $P^K(I_n)$, and the polynomial u_n has the following representation

$$u_n(t) = \sum_{k=1}^{K+1} u_{nk}\phi_{nk}(t), \tag{36}$$

where $u_{nk} = u_n(t_{nk})$, $t_{nk} = t_{n-1} + h_n\xi_k$, $k = 1, \ldots, K+1$. The basis for the polynomials v_n in $P^{K+1}(I_n)$ is obtained in the following manner. We construct the new family of polynomials as follows

$$\begin{cases} \psi_k(\tau) = \int_\tau^1 \phi_k(\xi)\, d\xi, & 0 \le \tau \le 1, \quad k = 1, \ldots, K+1, \\ \psi_0(\tau) = 1, & 0 \le \tau \le 1. \end{cases} \tag{37}$$

It is readily seen that the new family of polynomials $\{\psi_k : k = 0, \ldots, K+1\}$ is a basis of $P^{K+1}(0,1)$. The set of polynomials

$$\psi_{nk}(t) = \psi_k\Big(\frac{t - t_{n-1}}{h_n}\Big), \qquad t_{n-1} \le t \le t_n, \tag{38}$$

will be the desired basis for $P^{K+1}(I_n)$.

We now derive an equivalent set of equations for system (10). First, set $v_n(t) = \psi_{n0}(t) = 1$ in (10). We obtain

$$U_n = U_{n-1} + \int_{I_n} F(\hat{u}_n, t)dt. \tag{39}$$

Then set $v_n = \psi_{nk}$, $k = 1$, ..., $K+1$ in (10) and use the quadrature (33) to integrate exactly the polynomial $u_n \cdot \dot{v}_n$ with u_n given by (36). We obtain

$$U_n\psi_{nk}(t_n) - \int_{I_n} u_n\dot{\psi}_{nk}(t)dt \quad = \quad U_{n-1}\psi_{nk}(t_{n-1}) + \int_{I_n} F(\hat{u}_n, t)\psi_{nk}(t)dt.$$

But $\psi_{nk}(t_n) = \psi_k(1) = 0$, and $\psi_{nk}(t_{n-1}) = \psi_k(0) = \int_0^1 \phi_k(\zeta)d\zeta = \sum_{l=1}^{K+1} a_l\phi_k(\eta_l) = a_k$, then

$$\begin{aligned} \int_{I_n} u_n(t)\dot{\psi}_{nk}(t)dt &= \sum_{j=1}^{K+1} u_{nj} \int_{I_n} \phi_{nj}(t) \left(-\frac{1}{h_n}\dot{\psi}_{nk}(\frac{t - t_{n-1}}{h_n})\right) dt, \\ &= -\sum_{j=1}^{K+1} u_{nj} \int_0^1 \phi_j(\zeta)\phi_k(\zeta)d\zeta, \\ &= -\sum_{j=1}^{K+1} u_{nj} \sum_{i=1}^{K+1} a_j\phi_j(\eta_i)\phi_k(\eta_i), \\ &= u_{nk}a_k. \end{aligned}$$

Also for $k = 1, .., K+1$ we have

$$u_{nk} = U_{n-1} + \frac{1}{a_k} \int_{I_n} F(\hat{u}_n, t)\psi_{nk}(t)dt, \tag{40}$$

with $u_n(t) = \sum_{k=1}^{K+1} u_{nk}\phi_{nk}(t)$.

The equation (10) is equivalent to: $U_0 = x^0$, and for $n = 1, ..., N$

$$u_n(t) = U_{n-1} + \sum_{k=1}^{K+1} \frac{\phi_{nk}(t)}{a_k} \int_{I_n} F(\hat{u}_n, t)\psi_{nk}(t)dt$$

and

$$U_n = U_{n-1} + \int_{I_n} F(\hat{u}_n, t)dt,$$

where $\hat{u}_n$is the function defined by (9).

If we assume the integrals containing the nonlinear term F are evaluated with the $(K+1)$-point quadrature (33) which is exact for polynomials of degree $2K+1$, then (10) leads to the following numerical scheme : $U_0 = x^0$, and

$$u_{nk} = U_{n-1} + \frac{h_n}{a_k} \sum_{l=1}^{K+1} a_l F(\hat{u}_n, t_{nl}) \psi_k(\eta_l),$$

for all $k = 1, ..., K+1$ and

$$U_n = U_{n-1} + h_n \sum_{l=1}^{K+1} a_l F(\hat{u}_n, t_{nl}).$$

for all $n = 1, ..., N$, with $u_n(t) = \sum_{k=1}^{K+1} u_{nk} \phi_{nk}(t)$ and $\hat{u}_n$ is the function defined by (9).

7 Numerical tests

To validate the convergence results we consider two problems. In the first one, we consider a linear ordinary differential equation

$$\begin{cases} \dot{x}(t) = 2tx(t), & t \in [0,1], \\ x(0) = 1. \end{cases}$$

while in the second one, we take a nonlinear ordinary differential equation

$$\begin{cases} \dot{x}(t) = -2tx^2(t), & t \in [0,1], \\ x(0) = 1. \end{cases}$$

In this example the assumptions of Theorem 5.5 are satisfied with $B = 2$. In Figures 1 and 2 we consider the first example with one state-dependent impulse at the point 0.5 where $\alpha_1(x(0.5^-)) = x(0.5^-)$. Then we consider state-independent impulses. Two cases are considered for each example. In the first case we suppose that we have 1000 impulses in $\tau_i = \frac{7i\sqrt{2}}{10^4}$ and $\alpha_i = (-1)^{i+1}$. For the second case we suppose infinitely many state-independent impulses with an accumulation point at 1/3, with

$$\tau_i = \frac{1}{3} - \frac{1}{3^{i+1}} \text{ and } \alpha_i = \frac{1000}{2^i}.$$

For each case, we compute the logarithm of L^2-error ($ln(eL2)$) and the logarithm of the nodal error ($ln(eN)$) as functions of the logarithm of the step of the partition $ln(h)$.

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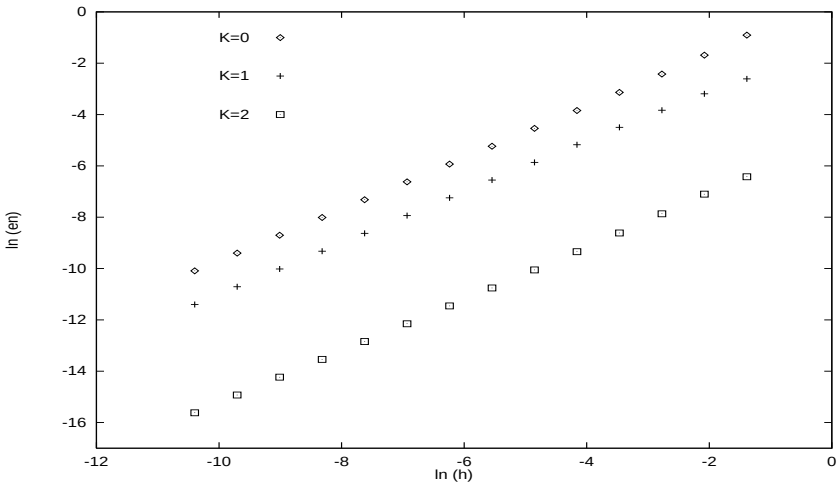


Figure 1: Nodal error with one state-dependent impulse

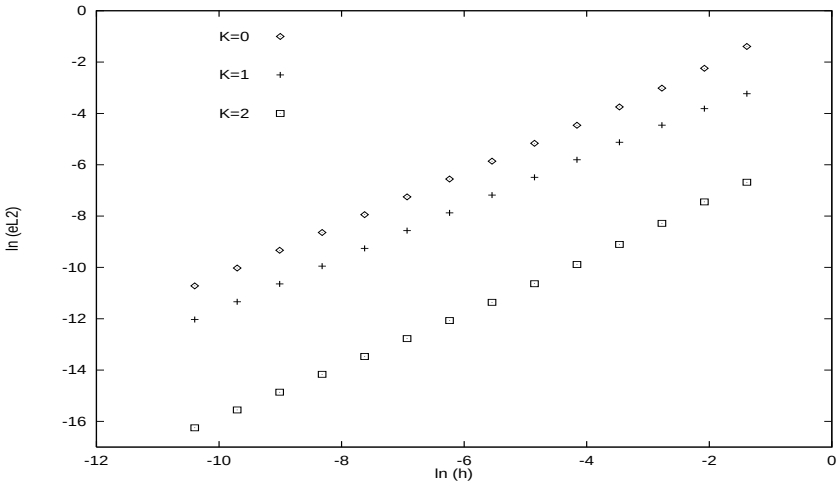


Figure 2: L^2-error with one state-dependent impulse

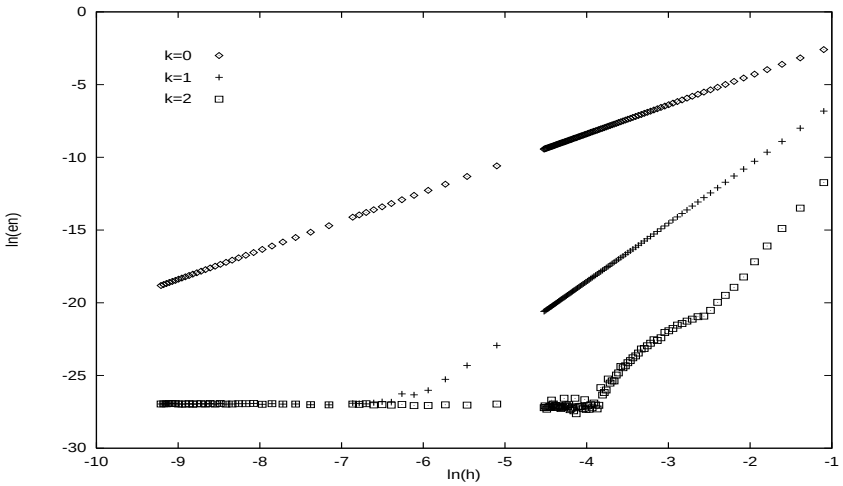


Figure 3: Nodal error for example 1 with 1000 impulses

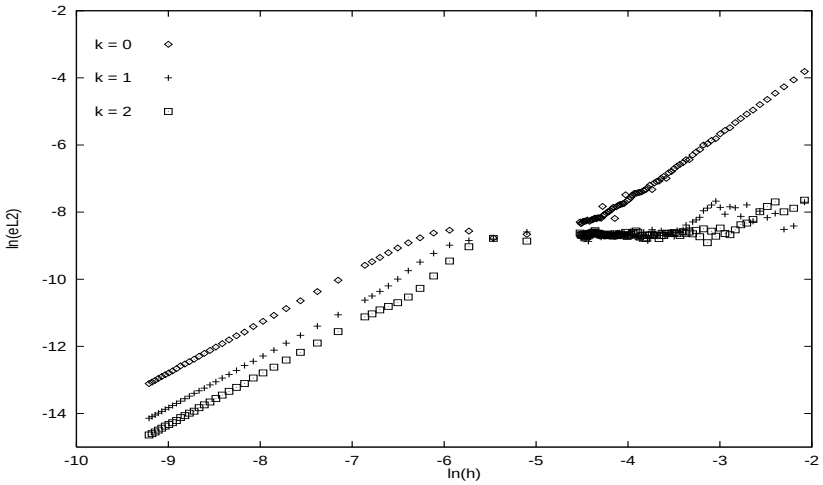


Figure 4: L^2-error for example 1 with 1000 impulses

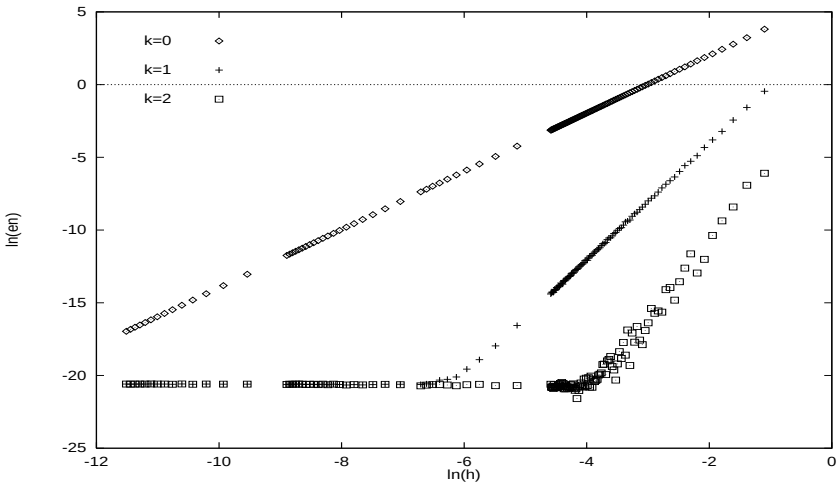


Figure 5: Nodal error for example 1 with infinitely many impulses

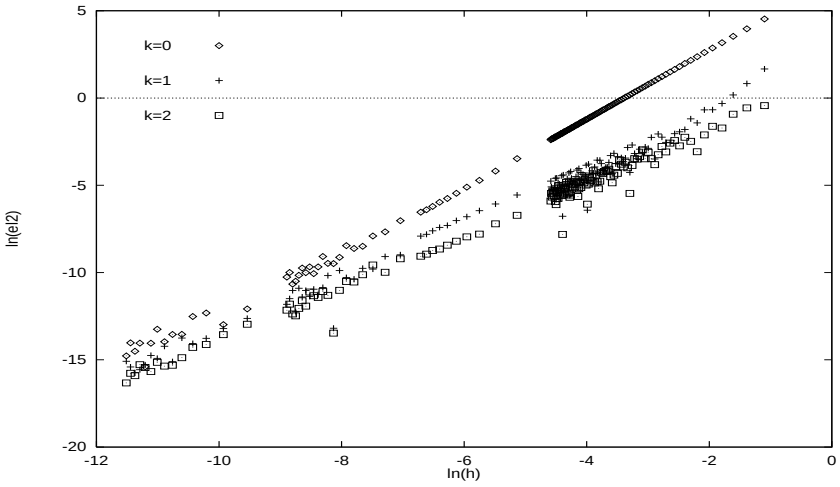


Figure 6: L^2-error for example 1 with infinitely many impulses

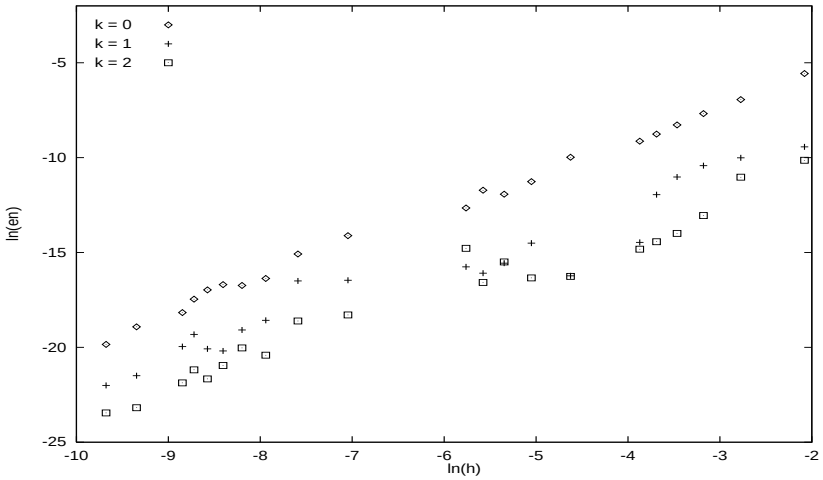


Figure 7: Nodal error for example 2 with 1000 impulses

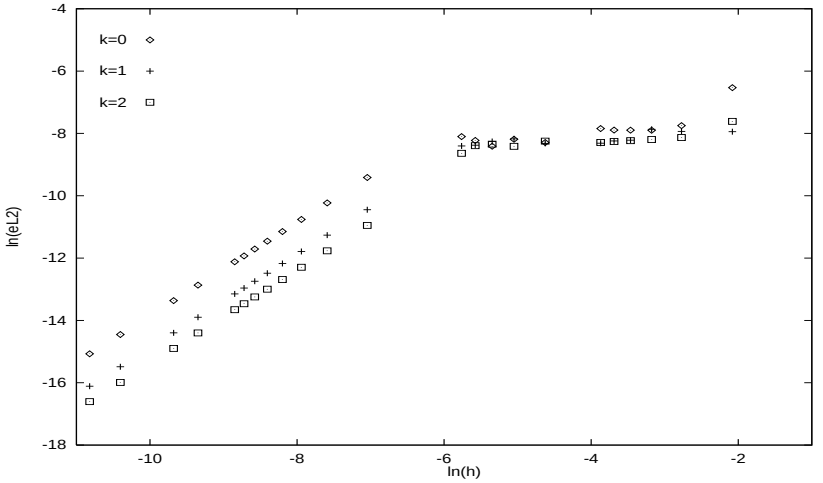


Figure 8: L^2-error for example 2 with 1000 impulses

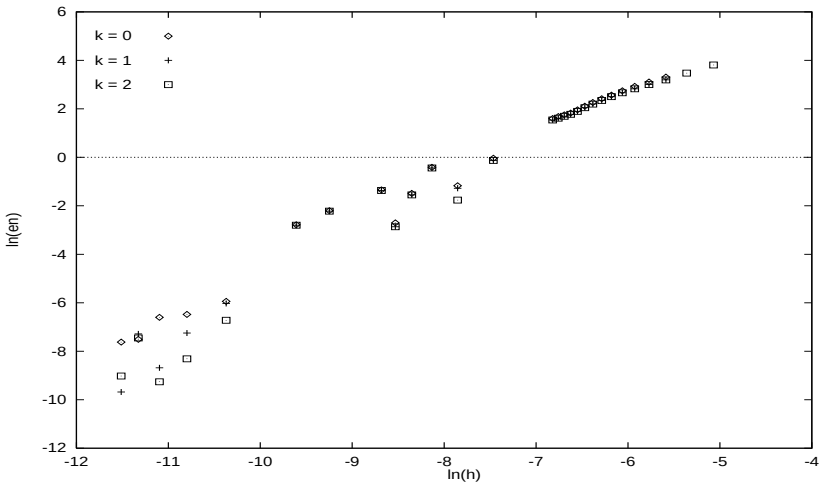


Figure 9: Nodal error for example 2 with infinitely many impulses

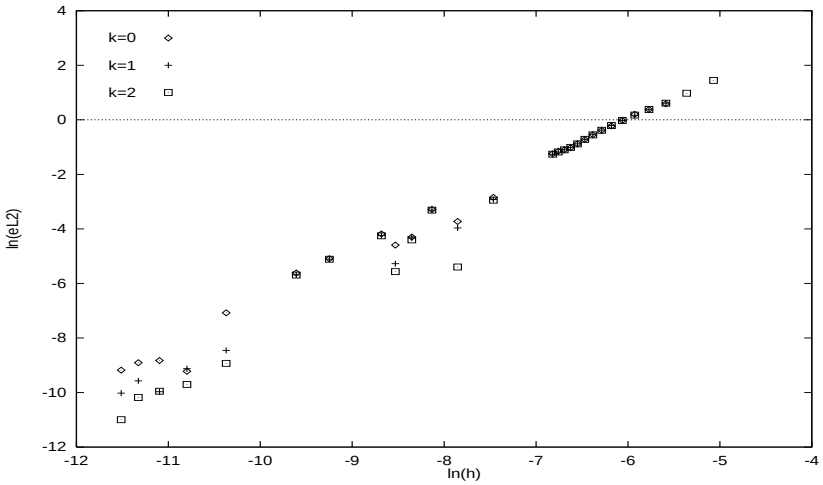


Figure 10: L^2-error for example 2 with infinitely many impulses

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The Newton Method for Hammerstein Equations

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Abstract

We study the convergence of Newton's method to a solution of a nonlinear integral equation of the Hammerstein type and second kind. We analyse the convergence of the method from a point of view different to traditional ones. We localize a solution of a particular integral equation and it is approximated directly by Newton's method. Finally, some numerical tests are provided, where nonlinear integral equations of the Hammerstein type with degenerated kernels are considered.

Keywords: nonlinear equations in Banach spaces, Newton's method, semilocal convergence theorem, recurrence relations, integral equation of the Hammerstein type.
Classification A.M.S. 1991: 65J15, 47H17.

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1 Introduction

In this paper, we study a particular nonlinear integral equation of the Hammerstein type and second kind:

$$x(s) = \ell(s) + \int_a^b \mathcal{K}(s,t,x(t))\,dt, \tag{1}$$

where $\mathcal{K}(s,t,x(t)) = K(s,t)[x(t)^{1+p} + x(t)^2]$, $p \in [0,1]$, and K is continuous and non-negative in $[a,b]\times[a,b]$, and ℓ is a continuous function such that $\ell(s) > 0$, $s \in [a,b]$.

This type of integral equation has two interesting aspects for us. Firstly, if we use in the study of (1) the usual procedure, the method of successive approximations, we need that the kernel $\mathcal{K}(s,t,x(t))$ is Lipschitz continuous in its third argument (see [3]). Observe that (1) does not satisfy this condition. Moreover, in this paper, contrary to Davis, we

do not consider a homogenous situation. Secondly, it is known (see [4]) that by means of the study of Newton's method we can locate domains of existence and uniqueness of solutions for (1) if we consider:

$$F : \mathcal{D} \subseteq C[a,b] \to C[a,b], \quad \mathcal{D} = \{x \in C[a,b];\ x(s) > 0,\ s \in [a,b]\},$$

$$[F(x)](s) = x(s) - \ell(s) - \int_a^b K(s,t)[x(t)^{1+p} + x(t)^2]\,dt, \quad p \in [0,1], \tag{2}$$

and we set out $F(x) = 0$.

The Newton method is the most used iteration to solve those equations as a consequence of computational efficiency even less speed of convergence can be got. Attention will now be given to the conditions under which the Newton sequence $\{x_n\}$ defined, starting from some point $x_0 \in X$, by the algorithm:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geq 0, \tag{3}$$

if $(F'(x_n))^{-1} \equiv \Gamma_n$ exists for $n = 0, 1, 2, \ldots$, will converge to a solution x^* of the equation $F(x) = 0$.

Under the hypothesis of the Newton-Kantorovich theorem [6]:

($\mathbf{C}_1$) $\|\Gamma_0\| \leq \beta$,

($\mathbf{C}_2$) $\|x_1 - x_0\| \leq \eta$,

($\mathbf{C}_3$) $\|F''(x)\| \leq L$ in some closed ball $\overline{B(x_0, \rho)}$,

($\mathbf{C}_4$) $h = \beta\eta L \leq 1/2$,

the convergence of the Newton sequence $\{x_n\}$ implies the existence of the solution x^*. This theorem gives not only conditions for the existence of x^* but also information concerning the regions of existence and uniqueness of x^* and error bounds for the terms x_n of the Newton sequence as approximations to x^*.

Once the values of β and η are calculated, we only need to determine $\|F''(x)\|$ in a neighborhood of x_0. The original proof, given by Kantorovich [7], uses recurrence relations. In [6], a new proof is presented where the concept of majorant function is used.

In [8], it is given another proof where majorant sequences are used and changes condition **($\mathbf{C}_3$)** by

$$\|F'(x) - F'(y)\| \leq L\|x - y\|, \quad L \geq 0, \tag{4}$$

where x, y are in an open convex domain $\Omega \subseteq X$. Other authors ([1], [2], [5]) consider the following generalization of the previous conditions:

$$\|F'(x) - F'(y)\| \leq L\|x - y\|^p, \quad L \geq 0, \quad p \in [0,1], \tag{5}$$

where x, y are in an open convex domain $\Omega \subseteq X$. Observe that if $p = 1$, (5) is reduced to (4).

However, operator (2) satisfies neither $(\mathbf{C}_3)$ nor (4) and (5) in $\mathcal{D}$.

For (2), we have:

$$[F'(x)y](s) = y(s) - \int_a^b K(s,t)[(1+p)x(t)^p + 2x(t)]y(t)\,dt, \tag{6}$$

$$[F''(x)yz](s) = -\int_a^b K(s,t)[(1+p)px(t)^{p-1} + 2]z(t)y(t)\,dt,$$

and

$$\|F''(x)\| \leq A\|x\|^{p-1} + B, \quad x \in \mathcal{D}, \tag{7}$$

where $A = (1+p)pM$, $B = 2M$ and $M = \max\limits_{s\in[a,b]} \int_a^b |K(s,t)|\,dt$ for the max-norm.

The interest of this paper is to prove the convergence of the Newton method for operators with second Fréchet derivative satysfying condition (7). For this, we use a new technique where a system of recurrence relations is constructed, and starting from them, the semilocal convergence of Newton's method is proved and a domain of existence of solutions of $F(x) = 0$, where F is given by (2), is also provided. After that, a domain of uniqueness of solutions for $F(x) = 0$ is obtained.

Finally, we consider a particular integral equation of type (1) and the domains of existence and uniqueness of solution are calculated. We also approximate its solution by the direct application of the Newton method. We finish with different numerical tests where integral equations of type (1) are considered.

Throughout the paper we denote

$$\overline{B(x,r)} = \{y \in X; \|y - x\| \leq r\} \quad \text{y} \quad B(x,r) = \{y \in X; \|y - x\| < r\}.$$

2 Localization of the solution

Firstly, we locate a starting point x_0 for Newton's iteration, so that $\Gamma_0 = F'(x_0)^{-1}$ exists. So, from (6), it follows that

$$\|I - F'(x)\| = ((1+p)\|x^p\| + 2\|x\|)M.$$

By the Banach lemma, if $((1+p)\|x^p\| + 2\|x\|)M < 1$, we have

$$\|F'(x)^{-1}\| \leq \frac{1}{1 - ((1+p)\|x^p\| + 2\|x\|)M}.$$

Therefore, $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(C[a,b], C[a,b])$ exists for some $x_0 \in \mathcal{D}$, where $\mathcal{L}(C[a,b], C[a,b])$ is the set of bounded linear operators from $C[a,b]$ into $C[a,b]$, if

$$((1+p)\|x_0^p\| + 2\|x_0\|)M < 1. \tag{8}$$

Moreover,

$$\|\Gamma_0\| \leq \frac{1}{1-((1+p)\|x_0^p\|+2\|x_0\|)M} = \beta.$$

Furthermore, $\|F(x_0)\| \leq \|x_0-\ell\| + \left(\left\|x_0^{1+p}\right\| + \|x_0^2\|\right) M$, according to the definition of the operator F, and, in consequence,

$$\|x_1 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \frac{\|x_0-\ell\| + \left(\left\|x_0^{1+p}\right\| + \|x_0^2\|\right) M}{1-((1+p)\,\|x_0^p\|+2\|x_0\|)M} = \eta.$$

The next aim is to prove the convergence of sequence (3) to a solution of the equation $F(x)=0$, where the operator F is defined in (2), and to obtain the domains of existence and uniqueness of solution. For, the first, it suffices to see that (2) is a Cauchy sequence. Firstly, we introduce the following auxiliary equation in t:

$$[2-3\beta t(A(\|x_0\|-t)^{p-1}+B)]t - 2[1-\beta t(A(\|x_0\|-t)^{p-1}+B)]\eta = 0, \quad p\in[0,1], \quad (9)$$

and we assume that it has one positive root less than $\|x_0\|$ at least. The smallest root satisfying this condition is denoted by R (i. e.: $R<\|x_0\|$).

Secondly, for $f(x)=(1-x)^{-1}$, we observe that

$$\eta\sum_{i=0}^{n}\left(\frac{\alpha}{2}f(\alpha)\right)^i = \eta\frac{1-\left(\frac{\alpha}{2}f(\alpha)\right)^{n+1}}{1-\frac{\alpha}{2}f(\alpha)} < \frac{\eta}{1-\frac{\alpha}{2}f(\alpha)} = R \quad (10)$$

if $\alpha=\beta R(A(\|x_0\|-R)^{p-1}+B)<2/3$. Therefore, it is supposed that $\alpha<2/3$.

Thirdly, for $x\in B(x_0,R)$, we have

$$\|I-\Gamma_0 F'(x)\| \leq \|\Gamma_0\|\|F'(x)-F'(x_0)\| < \alpha.$$

Since $\alpha<2/3<1$, then, by the Banach lemma, $F'(x)^{-1}$ exists and

$$\|F'(x)^{-1}F'(x_0)\| \leq f(\alpha).$$

Moreover, for (10), $\|x_1-x_0\|\leq\eta<R$, $x_1\in B(x_0,R)$ and $\Gamma_1=F'(x_1)^{-1}$ exists with $\|\Gamma_1 F'(x_0)\|\leq f(\alpha)$.

From Taylor's formula and (3) it follows that

$$F(x_1) = F(x_0)+F'(x_0)(x_1-x_0)+\int_{x_0}^{x_1}F''(x)(x_1-x)\,dx$$

$$= \int_0^1 F''(x_0+t(x_1-x_0))(x_1-x_0)(1-t)\,dt\,(x_1-x_0)^2,$$

and consequently

$$\|F(x_1)\| \leq \int_0^1 \|F''(x_0+t(x_1-x_0))\|(1-t)\,dt\,\|x_1-x_0\|^2$$

$$\leq \frac{1}{2}(A(\|x_0\|-R)^{p-1}+B)\|x_1-x_0\|^2 < \frac{1}{2}(A(\|x_0\|-R)^{p-1}+B)R\|x_1-x_0\|.$$

Thus

$$\|x_2-x_1\| \leq \|\Gamma_1 F'(x_0)\|\|\Gamma_0\|\|F(x_1)\| < \frac{\alpha}{2}f(\alpha)\|x_1-x_0\| < \eta < R$$

and

$$\|x_2-x_0\| \leq \|x_2-x_1\|+\|x_1-x_0\| < \left(1+\frac{\alpha}{2}f(\alpha)\right)\eta < R.$$

Hence we can follow the procdure because $x_2 \in B(x_0,R)$. We then assume, for $k = 1,2,\ldots,n-1$, that the next conditions are true:

(a) Γ_k exists and $\|\Gamma_k F'(x_0)\| \leq f(\alpha)$,

(b) $\|F(x_k)\| < \frac{1}{2}(A(\|x_0\|-R)^{p-1}+B)R\|x_k-x_{k-1}\|$,

(c) $\|x_{k+1}-x_k\| < \frac{\alpha}{2}f(\alpha)\|x_k-x_{k-1}\|$,

(d) $\|x_{k+1}-x_0\| < \eta\sum_{i=0}^{k}\left(\frac{\alpha}{2}f(\alpha)\right)^i < R.$

So, we can prove, by induction, that **(a)**–**(d)** also hold for $k=n$.

As $x_n \in B(x_0,R)$, then Γ_n exists and $\|\Gamma_n F'(x_0)\| \leq f(\alpha)$. Since $\|x_n-x_{n-1}\| < R$, we have

$$\|F(x_n)\| < \frac{1}{2}(A(\|x_0\|-R)^{p-1}+B)R\|x_n-x_{n-1}\| \tag{11}$$

and, by (10), we obtain

$$\|x_{n+1}-x_n\| \leq \|\Gamma_n F'(x_0)\|\|\Gamma_0\|\|F(x_n)\| < \frac{\alpha}{2}f(\alpha)\|x_n-x_{n-1}\| < \left(\frac{\alpha}{2}f(\alpha)\right)^n\|x_1-x_0\| < R.$$

Therefore

$$\|x_{n+1}-x_0\| \leq \|x_{n+1}-x_n\|+\|x_n-x_0\| < \eta\sum_{i=0}^{n}\left(\frac{\alpha}{2}f(\alpha)\right)^i < R.$$

Finally, it follows immediately that (3) is a Cauchy sequence. Indeed,

$$\|x_{n+m}-x_n\| \leq \|x_{n+m}-x_{n+m-1}\|+\|x_{n+m-1}-x_{n+m-2}\|+\cdots+\|x_{n+1}-x_n\|$$

$$\leq \sum_{i=n}^{n+m-1}\left(\frac{\alpha}{2}f(\alpha)\right)^i\|x_1-x_0\| \leq \frac{1-\left(\frac{\alpha}{2}f(\alpha)\right)^m}{1-\frac{\alpha}{2}f(\alpha)}\left(\frac{\alpha}{2}f(\alpha)\right)^n\eta$$

and consequently $\{x_n\}$ is a Cauchy sequence, since $\alpha < 2/3$. Thus, $\{x_n\}$ converges to a limit x^* such that $F(x^*)=0$ by letting $n\to\infty$ in (11).

So, the folowing semilocal convergence theorem is given for the Newton method when it is applied to operator (2).

Theorem 2.1 *We suppose that condition (8) is satisfied and let R be the smallest positive root of equation (9) which is less than $\|x_0\|$. Assume that R exists, $\alpha = \beta R(A(\|x_0\| - R)^{p-1} + B) < 2/3$, where $p \in [0,1]$,*

$$A = (1+p)pM, \quad B = 2M, \quad \beta = \frac{1}{1 - ((1+p)\|x_0^p\| + 2\|x_0\|)M},$$

and $\overline{B(x_0, R)} \subseteq \mathcal{D}$. Then the Newton method (3) converges to a zero x^* of operator (2).

3 Uniqueness of the solution

Now, we provide a result on the uniqueness of the zero x^* of operator (2).

Theorem 3.1 *Under the hypotheses of the last theorem, we can guarantee the uniqueness of x^* in $\overline{B(x_0, R)}$. Moreover, x^* is unique in $\mathcal{D}_0 = B(x_0, r) \cap \mathcal{D}$, where r is the smallest positive root of the equation:*

$$(1+p)pA\beta\|x_0\|^{p-1}(t^2 - R^2) - 2A\beta(t^{1+p} - R^{1+p}) + (1+p)p(t-R)(B\beta(t+R) - 2) = 0,$$

provided that r exists and $r > R$.

Proof. Firstly, we prove that x^* is unique in $\overline{B(x_0, R)}$. It is supposed that z^* is another zero of (1) in $\overline{B(x_0, R)}$. So, from the approximation

$$0 = \Gamma_0[F(z^*) - F(x^*)] = \left[\int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\, dt\right](z^* - x^*) = P(z^* - x^*),$$

it follows that if the operator $P = \int_0^1 \Gamma_0 F'(x^* + t(z^* - x^*))\, dt$ is invertible, then $z^* = x^*$. Then, by the Banach lemma, we only have to prove taht $\|I - P\| < 1$. Indeed,

$$\|I - P\| \leq \|\Gamma_0\| \int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\|\, dt$$

$$\leq \beta \int_0^1 \int_0^1 \|F''(x_0 + s(x^* + t(z^* - x^*) - x_0))\|\, ds\, \|x^* + t(z^* - x^*) - x_0\|\, dt$$

$$< \beta(A(\|x_0\| - R)^{p-1} + B) \int_0^1 ((1-t)\|x^* - x_0\| + t\|z^* - x_0\|)\, dt$$

$$\leq \beta(A(\|x_0\| - R)^{p-1} + B)R = \alpha < 2/3 < 1.$$

Secondly, we assume that z^* is in $\mathcal{D}_0 = B(x_0, r) \cap \mathcal{D}$. Similarly to the above, we have

$$\|I - P\| \leq \|\Gamma_0\| \left\| \int_0^1 F'(x^* + t(z^* - x^*)) - F'(x_0)\, dt \right\|$$

$$\leq \beta \left\| \int_0^1 \int_0^1 F''(x_0 + s((1-t)(x^* - x_0) + t(z^* - x_0)))\, ds\, ((1-t)(x^* - x_0) + t(z^* - x_0))\, dt \right\|$$

$$< A\beta \int_0^1 \int_0^1 \|(x_0 + s((1-t)(x^* - x_0) + t(z^* - x_0))\|^{p-1}\, ds\, (R + t(r-R))\, dt$$

$$+B\beta \int_0^1 \int_0^1 (R + t(r-R))\, ds\, dt$$

$$\leq A\beta \int_0^1 \int_0^1 \left(\|x_0\|^{p-1} - s^{p-1} \|(1-t)(x^* - x_0) + t(z^* - x_0)\|^{p-1} \right) ds\, (R + t(r-R))\, dt$$

$$+B\beta \frac{r+R}{2}$$

$$< A\beta \int_0^1 (R + t(r-R)) \left(\|x_0\|^{p-1} - \frac{1}{p}(R + t(r-R))^{p-1} \right) dt + B\beta \frac{r+R}{2}$$

$$\leq A\beta \left(\frac{r+R}{2} \|x_0\|^{p-1} - \frac{r^{1+p} - R^{1+p}}{(1+p)p(r-R)} \right) + B\beta \frac{r+R}{2} = 1$$

if $r > R$. ■

4 Application

Next, we solve a nonlinear integral equation of type (1) by direct application of Newton's method. It is considered that K is a degenerated kernel, which is considered by other authors (see [3]), i. e., the kernel K is such that $K(s,t) = S(s)T(t)$, so that the problem in infinite dimension can be solved.

Let be the following particular integral equation of type (1):

$$x(s) = 1 + s \int_0^1 t^8 [x(t)^{3/2} + x(t)^2]\, dt. \tag{12}$$

First of all, we calculate the domains of existence and uniqueness of solution. If we choose $x_0(s) = 1$ for theorem 2.1, we have

$$A = 1/12, \quad B = 2/9, \quad \beta = 18/11, \quad \eta = 4/11.$$

Equation (9) is then

$$\frac{1}{242} \left(-176 + \left(548 + \frac{24}{\sqrt{1-t}} \right) t - \left(264 + \frac{99}{\sqrt{1-t}} \right) t^2 \right) = 0$$

The smallest positve root and less than $\|x_0\| = 1$ is $R = 0.4289\ldots$. So $\alpha = 0.2333\ldots$ and the hypotheses of theorem 2.1 hold. In consequence, (12) has a solution x^* in

$$\{u \in C[0,1];\ \|u-1\| \leq 0.4289\ldots\}.$$

Following theorem 3.1, x^* is unique in

$$\{u \in C[0,1];\ \|u-1\| < 5.3598\ldots\} \cap \mathcal{D}.$$

For (12), the operators F and F', given respectively by (2) and (6), are

$$[F(x)](s) = x(s) - 1 - s\int_0^1 t^8[x(t)^{3/2} + x(t)^2]\,dt, \quad p \in [0,1],$$

$$[F'(x)y](s) = y(s) - s\int_0^1 t^8[(3/2)x(t)^{1/2} + 2x(t)]y(t)\,dt.$$

If it is supposed that $(F'(x))^{-1}$ exists, then $y(s) = (F'(x))^{-1}w(s)$. So, we consider,

$$y(s) = w(s) + sI,$$

where

$$I = \int_0^1 t^8[(3/2)x(t)^{1/2} + 2x(t)]y(t)\,dt.$$

If the last equality is multiplied by $s^8[(3/2)x(s)^{1/2} + 2x(s)]$ and integrated between 0 and 1, we obtain

$$I = \frac{\int_0^1 s^8[(3/2)x(s)^{1/2} + 2x(s)]w(s)\,ds}{1 - \int_0^1 s^9[(3/2)x(s)^{1/2} + 2x(s)]\,ds}$$

provided that $\int_0^1 s^9[(3/2)x(s)^{1/2} + 2x(s)]\,ds \neq 1$. Therefore

$$y(s) = (F'(x))^{-1}w(s) = w(s) + s\,\frac{\int_0^1 t^8[(3/2)x(t)^{1/2} + 2x(t)]w(t)\,dt}{1 - \int_0^1 t^9[(3/2)x(t)^{1/2} + 2x(t)]\,dt}.$$

The direct application of Newton's method is then

$$x_{n+1}(s) = x_n(s) - (F'(x_n))^{-1}F(x_n)(s) = 1 + s\,\frac{A_n - B_n + D_n}{1 - C_n},$$

where

$$A_n = \int_0^1 t^8[x_n(t)^{3/2} + x_n(t)^2]\,dt, \quad B_n = \int_0^1 t^8[(3/2)x_n(t)^{1/2} + 2x_n(t)]x_n(t)\,dt,$$

$$C_n = \int_0^1 t^9[(3/2)x_n(t)^{1/2} + 2x_n(t)]\,dt, \quad D_n = \int_0^1 t^8[(3/2)x_n(t)^{1/2} + 2x_n(t)]\,dt.$$

We start at $x_0(s) = 1$ and the approximated solution is $x^*(s) = 1 + 0.367498s$ (see table 1).

Note that the approximated solution x^* lies within the existence domain of solution obtained above (see figure 1).

i	$x_i(s)$
1	$1 + 0.341880s$
2	$1 + 0.367558s$
3	$1 + 0.367498s$

Table 1: Approximated solution of (12).

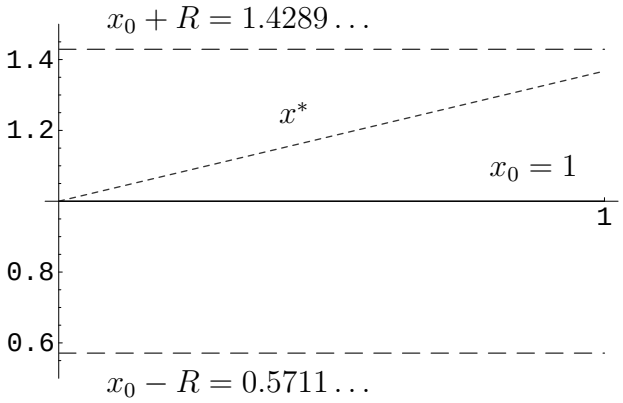


Figure 1: Graph of the approximated solution of (12)

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